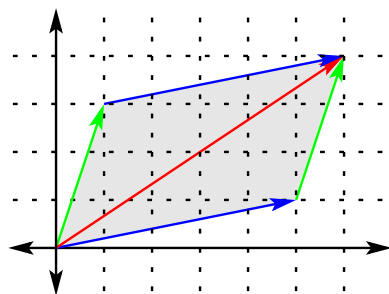


Here's a little piece about the standard form of the equation of a line.

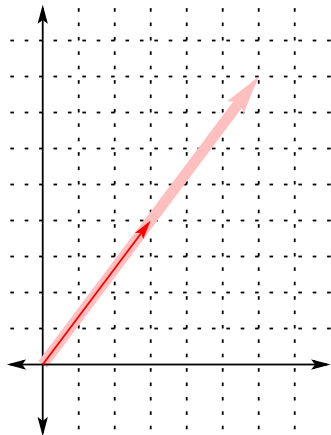
We should already be familiar with the *slope-intercept* form,  $y = mx + b$ , and the *point-slope* form,  $y - y_1 = m(x - x_1)$ . Notice that both of these forms depend on the existence of a *slope*,  $m$ ; thus, vertical lines cannot be expressed in either of these forms of the equation of a line, because such lines have no slope (it's not that they have a slope of 0; it's that there is no number at all which represents the slope of a vertical line). This brings us to the *standard* form of the equation:  $ax + by = c$ . As mentioned in class, any line in the plane (including vertical lines) may be expressed in standard form. The standard form does not require a line to have a slope.

Before we get into more interesting properties of the standard form, let's talk about *vectors*. We may think of a vector as an arrow drawn in the plane. It indicates a direction and a length. Below, see a picture of the vectors  $(5,1)$ ,  $(1,3)$ . Also pictured is their sum,  $(5,1)+(1,3)=(6,4)$ . The sum is calculated by *component-wise addition*: we add the  $x$ -coordinates 5 and 1 to get the new  $x$ -coordinate 6; then, we add the  $y$ -coordinates 1 and 3 to get the new  $y$ -coordinate 4. This is also called vector addition.

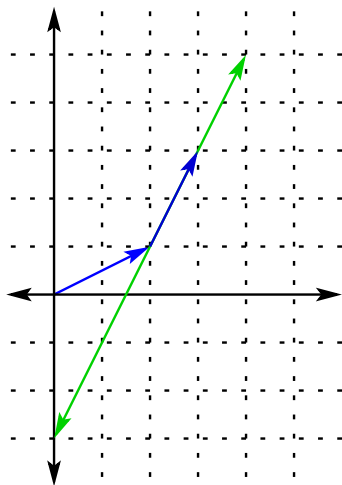


Note that  $(5,1)$  and  $(1,3)$  form edges of a parallelogram, and their sum forms the diagonal of that parallelogram. Basically this says that to *see* the sum of two vectors, just move one of them (without rotating it at all) so that it begins where the other one ends. It doesn't matter which one you move; you get the same answer either way.

People who talk about vectors sometimes use the word “scalar” to describe an *undirected* quantity, i.e., just a number. That may be because such numbers are often used in scaling transformations, or changes of scale. For example, if I want to make the vector  $(3,4)$  twice as long, I can talk about  $2(3,4)$ , which is a “scalar multiple” of  $(3,4)$ . The way we calculate a scalar multiple is just to multiply all the coordinates of the vector by the scalar; that is,  $2(3,4)=(6,8)$ . Find  $(3,4)$  and  $(6,8)$  plotted below.



Now, back to lines. A line may be described by a position vector and a direction vector. The position vector simply describes a single point on the line; then, the direction vector says which direction the line goes from there. Here we have a picture of the line (green) with position vector (2,1) (light blue) and direction vector (1,2) (dark blue).



To reach any point on the line, we first move to the base point (2,1); then we move as far as we like in the (1,2) direction. That is, any point on this line is of the form  $(2, 1) + t(1, 2)$  for some real number  $t$ . Here,  $t$  is the number of (1,2) steps we take. For positive  $t$ , we move forward along the line; for negative  $t$ , backward.

Let's look a little more closely at the points on this line. They are of the form  $(2, 1) + t(1, 2)$ . Note that  $t$  is a scalar, so we can multiply it through the (1,2) vector to get the equivalent form  $(2, 1) + (t, 2t)$ . Adding these two vectors, we get the (also) equivalent form  $\boxed{(2 + t, 1 + 2t)}$ . All of these descriptions of the line are called *parametric*, meaning they involve some "parameter" (in this case  $t$ ). It might be nice to get a *nonparametric* expression for the line; i.e., to write a description of the line that does not use  $t$ . Basically, this means getting back to an equation of the line in terms of  $x$  and  $y$ . We can see that the  $x$ -coordinate in the box is  $2 + t$ , so we have  $x = 2 + t$ . Likewise,  $y = 1 + 2t$ . In order to get rid of  $t$ , we should multiply one or more of these equations by a number and subtract so that the  $t$ 's all cancel. The two equations we have for  $x$  and  $y$  are marked below with a  $\star$ :

$$\begin{aligned} \star x &= 2 + t && \text{(Multiply both sides by 2)} \\ 2x &= 4 + 2t \\ \star y &= 1 + 2t && \text{(Subtract from the previous line)} \end{aligned}$$

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$$2x - y = 3$$

Hey, that looks like a standard form equation of the line! Great. But you could probably have found a standard form equation for this line much faster without using vectors, right? You'd find two points on the line—say, (2,1) and (3,3)—then find their slope, write the point-slope form of the equation, and convert it into standard form. Or maybe you'd do something else. So why talk about vectors and such?

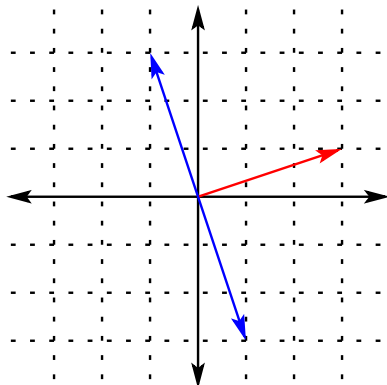
There's another operation we can do with vectors, called the dot product. It goes like this:  $(a, b) \cdot (c, d) = ac + bd$ . That is, the dot product of two vectors is a scalar. For example:  $(3, 7) \cdot (1, 0) = (3)(1) + (7)(0) = 3$ ;  $(3, 2) \cdot (1, -2) = (3)(1) + (2)(-2) = -1$ ; and  $(1, 2) \cdot (2, -1) = (1)(2) + (2)(-1) = 0$ . The dot product has something to do with how far two vectors travel in the same direction. Without

getting too precise: notice that the angle between  $(3, 7)$  and  $(1, 0)$  is acute, and their dot product is positive; the angle between  $(3, 2)$  and  $(1, -2)$  is obtuse, and their dot product is negative; the angle between  $(1, 2)$  and  $(2, -1)$  is a right angle, and their dot product is 0.

In general, two vectors are perpendicular (or *orthogonal*) if their dot product is 0. For instance, if the vector  $(x, y)$  is orthogonal to  $(3, 1)$ , then

$$\begin{aligned}(3, 1) \cdot (x, y) &= 0 \\ 3x + y &= 0\end{aligned}$$

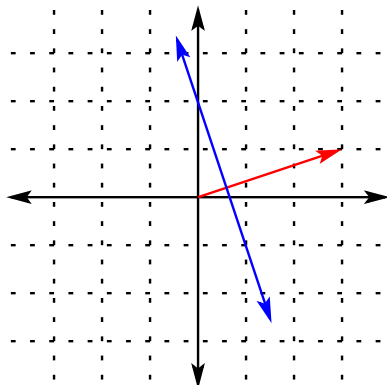
We see that the dot product quickly turns into a standard form equation of the line. We may rearrange that equation as  $y = -3x$ ; let's see a graph of the line (blue) and the vector (red) together:



By setting the dot product of  $(3,1)$  with  $(x, y)$  equal to zero, we obtain the line through the origin, perpendicular to  $(3,1)$ . If we set that dot product to a positive number, rather than zero, the line moves in the direction the vector points. For example, suppose we set it to 2 instead of 0:

$$\begin{aligned}(3, 1) \cdot (x, y) &= 2 \\ 3x + y &= 2\end{aligned}$$

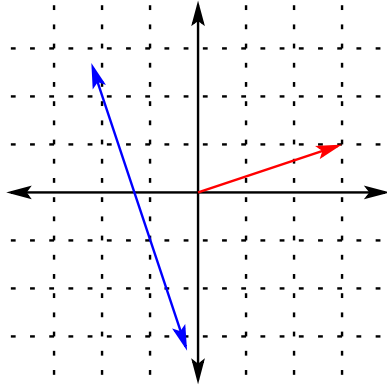
This is equivalent to  $y = -3x + 2$ , pictured below:



Notice how the line was translated in the direction that the vector points. If we instead set the right hand side equal to a negative number, such as -4, we obtain:

$$\begin{aligned}(3, 1) \cdot (x, y) &= -4 \\ 3x + y &= -4\end{aligned}$$

This is equivalent to  $y = -3x - 4$ , pictured below:

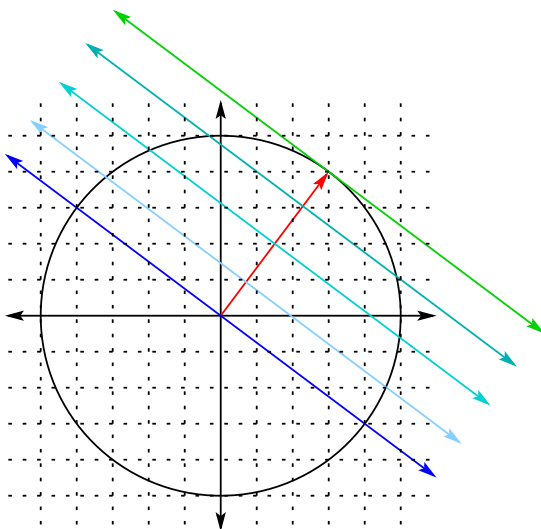


See how the line moved opposite the direction indicated by the vector.

Recall one of your practice midterm problems: find an equation for the line tangent to the circle  $x^2 + y^2 = 25$  at the point  $(3,4)$ . Note that the vector  $(3,4)$  is a radius of that circle. The line  $3x + 4y = 0$  is the line through the *origin* which is perpendicular to the vector  $(3,4)$ . We want to move this line (without changing its direction) so that it goes through the point  $(3,4)$ , instead. So, we change the 0 in that equation into some other number, say  $k$ , such that  $(3,4)$  is a solution to the equation  $3x + 4y = k$ . That is, (substituting 3 for  $x$  and 4 for  $y$ )  $3^2 + 4^2 = k$ , or  $k = 25$ .

Thus, the line  $3x + 4y = 25$  is perpendicular to the  $(3,4)$  direction and passes through the point  $(3,4)$ ; hence, it is the required tangent line. In the picture below, see the vector  $(3,4)$  in red, together with the lines

$$\begin{aligned}
 3x + 4y &= 0 \quad (\text{blue}) \\
 3x + 4y &= 6.25 \\
 &\vdots \\
 3x + 4y &= 25 \quad (\text{green})
 \end{aligned}$$



The last thing I'll say about this is that it generalizes very nicely into higher dimensions. For example, the equation  $3x - 2y + z = 0$  is equivalent to the vector equation  $(3, -2, 1) \cdot (x, y, z) = 0$ , and describes a plane through the origin (in three dimensions) which is perpendicular to the vector  $(3, -2, 1)$ . You'll hear more about this kind of thing in a 3rd- or 4th-quarter calculus class.