This is the first in a series of papers I'll write this quarter. I call this feature Down the Rabbit Hole. Sometimes in class we will see a basic idea that looks perfectly innocent and self-contained, when really there is a whole branch of mathematics hiding behind it. I hope to give you a glimpse of a few of the secret truths in mathematics. And like Alice, or Neo, or whoever's metaphor we're chasing at the moment, you can choose whether to jump down the rabbit hole or not. You will not be tested on any material that appears in here (unless it would also appear in the course had I not written this).

Today, let's think about the midpoint formula. It may also help for you to be comfortable with "directed distance"—see p.0-9 in Section 0.2.

The midpoint formula seems innocuous enough: it says that the midpoint between (a, b) and (c, d) is

$$\left(\frac{a+c}{2}, \frac{b+d}{2}\right) = \frac{1}{2}(a+c, b+d) = \frac{1}{2}(a,b) + \frac{1}{2}(c,d).$$

Or as I like to say it, if you average the x-coordinates of two points, and average their y-coordinates, the resulting point is the average of the two points. Their midpoint.

Instead of taking an ordinary average, we could also take a *weighted* average. Weighted averages are used whenever the ingredients represent different proportions of the end result. For example, suppose you have 3 quiz scores: 8, 8, and 5. Here we could say that the only scores you got were 8 and 5, but that the 8 happened twice—which is to say, it should have twice the weight of the 5 score in determining your grade. Your average quiz score will be

$$\frac{8+8+5}{3} = \frac{2 \cdot 8 + 5}{3} = \frac{2}{3} \cdot 8 + \frac{1}{3} \cdot 5.$$

Numerically, that's 7. Check out 5,7, and 8 on the number line—you'll see that 7 is twice as far from 5 as it is from 8.

The same thing works for coordinates of points. Let's say we have the points (2,5) and (17,35). Instead of finding the midpoint, let's find another point somewhere in between. Let's try two parts (2,5) and three parts (17,35). That's five parts total, so the weighted average is

$$\frac{2 \cdot (2,5) + 3 \cdot (17,35)}{5} = \frac{(4,10) + (51,105)}{5} = \frac{(55,115)}{5} = (11,23).$$

If you check them out on a graph, you'll see that (11,23) is between (2,5) and (17,34), a little closer to (17,35) than to (2,5). Try it again on your own, using two parts (2,5) and one part (17,35).

So what's the theme here? What is shared by the following expressions?

$$\frac{1}{2} \cdot (a,b) + \frac{1}{2} \cdot (c,d)$$

$$\frac{2}{3} \cdot 8 + \frac{1}{3} \cdot 5$$

$$\frac{2}{5} \cdot (2,5) + \frac{3}{5} \cdot (17,35)$$

$$\frac{4}{7}(\text{ginger ale}) + \frac{2}{7}(\text{lemon juice}) + \frac{1}{14}(\text{grenadine}) + \frac{1}{14}(\text{simple syrup})$$

For starters, each of the four expressions above contains two or more terms; each term consists of a coefficient, such as $\frac{3}{5}$, and an ingredient, such as (17,35). These observations are not interesting. The interesting part, as I'm sure you'll agree, is that *the coefficients appearing in each expression always add up to one.*

The right mindset for reading mathematics is one of amused skepticism. For example, just a second ago, you should have chuckled to yourself, "<chuckle, chuckle> Brad says the coefficients all add up to one, does he? Well, I'm not falling for that! Just let me go check those numbers real quick..." You might even ask yourself *why* a weighted average should turn out this way. (They always do.)

Once you're convinced about the coefficients–add–to–one theory, you're ready for the next step: generalization. Let's get back to the idea of two points in the plane, (a, b) and (c, d). Rather than generate two particular coefficients, like $\frac{2}{5}$ and $\frac{3}{5}$, I'd like to think in more *general* terms. Let's call one of the coefficients t. Since the two coefficients have to add up to one, the other coefficient is $1 - t$. Be skeptical! Confirm that this actually does give us two coefficients that add to one, and then be slightly amazed. OK, I did say slightly.

For picky reasons that I'll keep to myself, I want $1 - t$ to be the coefficient of (a, b); thus, t is the coefficient of (c, d). So, the whole expression is

$$(1 - t) \cdot (a, b) + t \cdot (c, d).$$

This represents an *arbitrary* weighted average of (a, b) and (c, d)—an arbitrary point between them.

If you're reading this sentence and you haven't called me a liar, you're reading too fast. Go back there and try out my expression—pick specific numerical values for a, b, c, and d; then (while keeping those four numbers fixed) try out a few different values of t. Plot all the resulting points in the plane (make your two initial points, (a, b) and (c, d), a different color or something). What do you think about my statement that the expression represents an arbitrary point between (a, b) and (c, d)?

Of course, some readers might just shrug their way past the preceding paragraph and expect me to answer my own question here. I won't do it! Are you really so eager to let someone else tell you what to think? Go on, try the experiment if you haven't already. The more different values of t you use, the better. Make sure you try 0, 1, a number between 0 and 1, a number bigger than 1, and a number less than 0.

Usually, when mathematicians talk about this subject, they use the formal terms "affine combination" and "convex combination". If you feel so inclined, look them up at www.mathworld.com. Hm. Actually, I felt inclined, and I looked. Not what I expected to find. Well, let me just say that mathworld is usually a pretty good site, even if it was a letdown this time.

By now, you've probably forgotten that I mentioned the directed distance formulas. Now that I've reminded you, you're probably wondering what they have to do with this discussion. (I'm such a mindreader!) Well, let's start with statement 1 on p.0-9:

"The directed distance from a to b is $b - a$."

This is true because if we start at the number a and add the directed distance $b - a$, we arrive at the number $a + (b - a) = b$. That is, $b - a$ is the stuff we should add to a in order to reach b. When we're just dealing with numbers, this is a little...useless. For the topic at hand, I mean. Let's bump it up a dimension:

"The directed distance from (a, b) to (c, d) is $(c, d) - (a, b)$."

Do you believe this? What happens if we start at (a, b) and add the quantity $(c, d) - (a, b)$? Replace a, b, c, and d with specific numbers if it helps, until you've made sense of the 2-dimensional notion of directed distance.

Some things to ponder:

- Why did I call this "bumping it up a dimension"?
- In what sense are these two notions of directed distance *the same thing*?
- What could I possibly mean by a statement like, "The directed distance from P to Q is $(1, -2)$"?

And now, starting with the generalized concept of midpoint, we will derive what is known as the vector form of the equation of a line:

$$\begin{aligned} (1-t)\cdot(a,b)+t\cdot(c,d) &= 1\cdot(a,b)-t\cdot(a,b)+t\cdot(c,d) \\ &= (a,b)+t\cdot(c,d)-t\cdot(a,b) \\ &= (a,b)+t[(c,d)-(a,b)] \end{aligned}$$

Notice, in the square brackets in the last expression above, we have the directed distance from (a,b) to (c,d). It determines a direction of travel. By altering the value of t in the expression $(a,b)+t[(c,d)-(a,b)]$, we adjust how far we go in that direction. With $t=0$, we don't move at all; we just stay at the base point (a,b). With $t=1$, we start at (a,b) and move exactly the directed distance needed to reach (c,d). Different values of t take us to different destinations, all along the same line.

That's as far as we'll go today, in full mathematical detail. There are a few places we could go from here. One is Vector Calculus, which is used in a ridiculous number of areas. The word "vector" does not appear in the index of our textbook, but basically it means "directed distance". Vectors are very helpful for tracking the movement of objects. If the slope-intercept equation of a line tells us where the train tracks are, then the vector form of the equation of a line gives us a train schedule. Math 21D treats this subject in detail. Applications arise in the Physics 9 series and throughout engineering.

We could also dive into Convex Set Theory (remember "convex combination"?). Where does convex set theory turn up? Well, did you know that there is a company in Davis that has been building *flying cars* for the past 20 years or so? Check them out at www.moller.com. Their engine design is based on the Reuleaux Triangle (which is also used by Mazda). It's a nice shape that shows up in convex set theory. It's also the basis for a drill bit that can drill a square hole. Convex set theory has something to say about submarines and the Challenger Disaster (the Columbia Disaster of my generation), and it explains why a painter needs more colors than just red, blue, and yellow. Of course, most of its results are a bit more mathy than these examples, but it's difficult to explain the appeal of such things in a blurb.

One last thing. The Shirley Temple recipe above calls for simple syrup, which is basically an amount of sugar dissolved in an equal volume of water, on the stove. Boil for a couple minutes, then chill. It keeps for a week or two.