This is the second of the Rabbit Hole papers. As before, you will not be tested on any material that appears in here, unless it would also appear in the course had I not written this.

Symmetry is one of my favorite ideas in all of mathematics. The idea of symmetry addresses repetition of structure; it appears most obviously in geometry, but also in logic and other areas.

Hey, if you haven't read it yet, check out "Godel, Escher, Bach: an Eternal Golden Braid," by Douglas Hofstadter (it was out of print last time I checked, but there may still be hope). Kurt Godel was a logician, the author of Godel's Incompleteness Theorem—a result of profound philosophical importance. Among other things, his theorem demonstrates that *we will never know all of mathematics*. M.C. Escher was an artist; we'll say more about him in a minute. Johann Sebastian Bach was...well come on now, if you've ever heard music, you must have heard Bach. Anyway, Hofstadter's book shows how self-reference appears in all their works. Except my description makes it sound dull. Perhaps you'd prefer Martin Gardner's review to mine. He says, "Every few decades an unknown author brings out a book of such depth, clarity, range, wit, beauty, and originality that it is recognized at once as a major literary event. [This] is such a work."

And now that I'm done selling somebody else's book, we can get back to symmetry. One way to understand the symmetries of planar figures is to ask, "How may I move the plane, without disrupting this figure?" For instance, if the figure in question is a circle, then you can rotate the entire plane about the circle's center as much as you like, and the circle will not be perturbed at all. Sure, it turns along with everything else—but regardless of how much it turns, it is always superimposed on its original position. It has perfect rotational symmetry; in a sense, it is immune to all rotations about its center. We also say that it is *preserved* by rotations about this center.

On page 7 of our text, we are introduced to the idea of "translating points in the plane". An example is given, in which a rhombus is translated by the directed distance (4,-2). ("Translated" means "moved without rotation", and when we speak of a "translation", we basically mean the process of translating all points in the plane by the same amount.) In the book's example, we see one rhombus, and then we see it with its duplicate. Since the two rhombi are not *perfectly superimposed*, we say that the original rhombus is NOT preserved by the given translation.

Before I continue with this extended tangent, I'd like to bring the conversation back to ideas relevant to our course. There was a problem on the first homework, on which you were to demonstrate via distance formula that the quadrilateral with vertices A(0,0), B(1,2), C(3,3), D(2,1) is a rhombus; that is, show that its four sides are all the same length. The assignment asks you to do it by using the distance formula, so that was the way to go about it for full credit, but transformational geometry gives us a much more elegant solution.

As far as you need to know for now, the big idea in transformational geometry is that there are transformations, called *motions* (or *isometries*), that preserve length. That is, any pair of points will be the same distance apart after the transformation as they were before it. Iso means *same*, as in *iso*sceles (geometry), *iso*tonic (chemistry), *iso*buttons (sewing), etc. Metry means distance, as in...meter. So *isometry* (pronounced eye-SAW-muh-tree) means "same distance". Oo, I love pretending I'm a linguist! What was I getting at? Oh yes—as it happens, these motions contain the idea of congruence. Geometric shapes are congruent (i.e. same shape and size) if and only if some motion superimposes them exactly. Did you ever learn Angle-Side-Angle or Hypotenuse-Leg or any others of those triangle congruence theorems in high school? Transformational geometry makes all that a bit easier...and it also works for circles, squares—in fact, anything you care to draw!

There are four kinds of motions. The *direct* motions are rotations and translations. They keep the page face-up. The *opposite* motions are reflections and glide reflections (basically you reflect across a line and then slide some distance along that same line). Opposite motions flip the plane over. If you want to prove that two things are congruent, just find a motion that takes one to the other. For instance, the translation by the directed distance (2,1) takes vertex A in our alleged rhombus to vertex D, and likewise it takes B to C. You can check this by adding (2,1) to (0,0) and to (1,2). Since it sends A to D and B to C, it also sends the line segment AB to the line segment DC. That is, segments AB and DC are congruent, or—drum roll, please—the *same length*! Also, the reflection across the line $y = x$ swaps B with D while leaving A and C fixed; thus, it sends AB to AD and BC to DC. That is, AB and AD are the same length; so are BC and DC. We now have that AD is the same length as AB is the same length as DC is the same length as BC (I'm no longer impersonating a linguist, am I). And, ta-da, we've shown that the given quadrilateral is a rhombus using incredibly basic computations (literally, the hardest computation was 2+1=3). There were other, more advanced ideas at work as well, but I really enjoyed not seeing a radical sign. Transformational geometry is not always so clean, but this little demo didn't even scratch the surface of its power.

Please, if the last few paragraphs were confusing, DRAW the rhombus, and read the paragraphs again.

Now let's get back off course. Let me ask you a question: if a circle is preserved by rotation about its center, then what kind of object is preserved by translation by a certain directed distance? Perhaps the simplest such object is a line parallel to the direction of translation. But this line has many other symmetries as well: it can be reflected over itself or any perpendicular line, rotated 180 degrees about any point on itself, or acted on in a few other ways, all without really *doing* anything to the line. In other words, it has many symmetries besides the one we wanted to exemplify.

See if you can come up with a figure that is preserved by the (4, -2) translation—but try to make a figure with as few other symmetries as possible. For example, an infinite row of appropriately placed copies of the letter S is preserved by the given translation, so it's an example of an object with that translational symmetry; however, it also has a certain rotational symmetry (can you see it?). You can do better than this example of mine.

The Dutch graphic artist M.C. Escher created a large collection of patterns exemplifying different kinds of symmetry in the plane. (He also did other stuff, but mathematicians go *crazy* over his symmetry work.) Check out www.mcescher.com and whatever you do, DON'T SKIP THE GALLERIES! The last of the six galleries deals specifically with symmetry.

I would also draw your attention to the Circle Limit pieces in the Recognition and Success gallery, for these contain symmetries in a *different kind of geometry* than the plane geometry most of us learned in high school. These are pictures of Hyperbolic Geometry, a very strange place indeed. Why bother with exotic geometries? Well, I get the impression that physicists are increasingly convinced: our universe is *not* like xyz-space (the three-dimensional version of the plane we always use). Our universe has a different *shape*. But that's another trip down the rabbit hole.

For now, I'd like to play with a few symmetries. Let's begin by identifying all the symmetries of the infinite row

... TTTTTTT ...

Notice that if I slide this row of T's left or right by the width of one T (or two, or three...), then the new image will be perfectly superimposed on the original. That is, the pattern is preserved by certain translations. It is also preserved by certain left/right reflections. What do you think about these rows?

1. ... DDDDDD ...
2. ... pdpdpd ...
3. ... bpbpbp ...

Row 1 is preserved by horizontal translations by an integer number of letter-widths, and also by reflections across the horizontal centerline. Can you find any other motions that preserve the row of D's?

You may notice that row 2 cannot be reflected without turning it into a sequence of b's and q's, but it is preserved by a 180-degree rotation centered on the gap between any two consecutive letters.

Row 3 is a little harder to see. If you just slide it one letter to the right, you have b's lining up on p's, so that's no good; translations must be by an even number of letter-widths. Reflecting across a horizontal line *almost* works, but not quite. But if you slide the row one letter-width to the right *and* reflect it across the horizontal centerline, then it fits perfectly. This is an example of a *glide reflection*.

I leave you with a little task: look through the M.C. Escher symmetry gallery, and try to find an example of a pattern with *glide reflection* symmetry. Look for other kinds of symmetry as well.

Oh...in case you were wondering, this kind of math is explored in Math 150B.