Today's topic: Functions With Wacky Continuity Issues (and a Fractal)

We've often used the intuition that a function is continuous if you can draw it in a single stroke. I'd like to give you some examples of very bizarre continuity behavior, where this intuition doesn't serve us well. I think these illustrate the murky depths you can get into when you look very closely at something. They point out the importance of holding onto an intuitive idea only to the extent that it is *correct*, and that a hunch, while useful, is not the same thing as a well-considered logical argument. Plus, I just think they're fascinating. As with the other Rabbit Holes, this constitutes somewhat advanced and very optional reading.

First, you should know what rational and irrational numbers are; see p.0-2 of the text. Second, you should know that the rational numbers and the irrational numbers are so incredibly common that *every* interval, no matter how short, contains at least one of each. In case you were wondering, [3,3] is not an interval, even though it's written in interval notation. It's just the number 3.

And, since I'm incapable of writing three paragraphs without an aside, it's time for...

Aside: yes, the rationals and irrationals are extremely common. There are infinitely many of each; moreover, they are *dense* (which means every interval contains one—or, if you prefer, no matter where you stand on the real line, and no matter how short your arms are, you can reach one). But there is something decidedly strange about them. It begins innocuously enough: between every two irrationals is at least one rational, and between every two rationals is at least one irrational. Now, suppose by analogy that these were students in a finite-sized class, and we could line them up with a boy between every pair of girls and vice-versa. They would be arranged GBGB...GB, with perhaps an extra boy at the beginning or an extra girl at the end. We would know that the number of girls in the class was nearly equal to the number of boys, with a difference of at most one. But this intuition does *not* extend to infinite sets. There are far more irrationals than rationals on the real line. As if that weren't strange enough, it turns out that the rational numbers, in all their infinite magnificence, account for 0% of the real numbers. The remaining 100% are all irrational. There are *overwhelmingly* more irrational numbers than rational. This kind of insanity can be explained by a proper investigation of infinity, or more accurately, infinit*ies*. There are at least as many kinds of infinity as there are integers; each of them is so overwhelming larger or underwhelmingly smaller than the next one that they always fall into this 0% vs. 100% kind of proportion. And here I'm talking about what are called *infinite cardinal numbers*. (They do not live in the set of real numbers, because every real number is finite.) This is different from the so-called "infinity" in the statement

$$\lim_{x\to\infty} f(x) = L,$$

in which the symbol ∞ does not actually represent an infinite number; it simply means that x grows beyond all restraints.

Ah, my tangent-craving has been satisfied. Back to strange functions. We'll start with the classic example of a function that is discontinuous everywhere:

$$f_1(x) = \begin{cases} 0, & \text{if } x \text{ is rational;} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

The graph of this thing is a little like two horizontal lines ($y = 0$ and $y = 1$), except each line is heavily perforated. Pick any real number c. This function is discontinuous at $x = c$ because, no matter how close x gets to c, there is always at least one rational number, and at least one irrational number, in the interval between x and c. That is, the function hops up and down between the two lines, never settling on just one value.

Now let's see a function that is continuous at the origin, but nowhere else:

$$f_2(x) = \begin{cases} x, & \text{if } x \text{ is rational;} \\ -x, & \text{if } x \text{ is irrational} \end{cases}$$

This is similar to the last one, except this time the two perforated lines ($y = x$ and $y = -x$) intersect at the origin. This is the first time we've seen a function that has a *point* of continuity, but not an *interval* of continuity. The reason it's continuous at the origin is that both parts approach the same number as x approaches zero.

We can generalize this example to make a function that is continuous at any finite collection of numbers we choose. Suppose we'd like a function to be continuous at 2, 4, and 12; we define the function like this:

$$f_3(x) = \begin{cases} (x-2)(x-4)(x-12), & \text{if } x \text{ is rational;} \\ -(x-2)(x-4)(x-12), & \text{if } x \text{ is irrational} \end{cases}$$

We can see that both parts approach zero for x values very close to 2, 4, or 12. For x values approaching any other number, the top and bottom pieces lie on opposite sides of the x-axis. Since the function hops between the two on every interval, this makes the function discontinuous at all numbers except 2, 4, and 12 (and at these numbers its limit is zero).

Our final example will be a function that is continuous at every irrational number, but discontinuous at every rational number. I won't write it using the left curly brace, because I'd have to put a little paragraph inside it, and that's just bad form. Let's call our function something original and sassy, like f_4. If x is *irrational*, then $f_4(x)$ is defined to be zero. If x is *rational*, then x can be written in the form p/q, where p and q are integers with no common factors, and q is positive. Basically I'm saying write x in lowest terms, and if x is negative, then keep the minus sign upstairs. Now, once we have $x = p/q$ in this nice form, we define $f_4(x) = 1/q$. Whatever the numerator of x is, turn it into one. Try graphing this on the interval [0,1]. You have to go by hand, because your calculator is not equipped for this kind of work. This is one of those times when a T-chart is helpful; try $x = \frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \ldots$ Note that $\frac{2}{4}$ is not in lowest terms. Anyway, pick a denominator q and systematically go through all possible numerators p, $0 < p < q$. Increase q and repeat. I can tell you that the graph looks a little triangular, and reminds me of anti-aircraft artillery at night.

One thing you might ask is why I was so picky about how we wrote x. If you don't simplify fractions, what are $f(1/2)$ and $f(2/4)$, and what does this do to our graph? Another interesting part is trying to figure out why this function is continuous at every irrational number and discontinuous at every rational number; I'll leave that to you.

Hm, the triangular nature of this graph got me thinking. There's something totally different, but way awesome, that you can do with your calculator. Here's a program that I wrote for my TI-81 back when it was still cutting-edge. Any of you with more current TI's will find that the same program (perhaps with updated syntax?) will go a lot faster. It runs forever, so you should just stop it when you get bored by pressing `ON`. It plots a *fractal* known as Sierpinski's Triangle. Fractals are mathematical objects of often hypnotic beauty; they frequently arise from very simple, self-referent generators. (Self-reference showed up in the second Rabbit Hole, remember?) Anyway, the program is on the next page.

```
Program:SIERP
:All-Off
:0→Xmin
:1→Xmax
:0→Ymin
:1→Ymax
:PT-Off(0,0)
:PT-Off(1,0)
:PT-Off(.5,1)
:.5→A
:1→B
:DispGraph
:Lbl 1
:Rand→Z
:((A+(Z>1/3)/2+(Z>2/3)/2)/2)→A
:((B+(Z>1/3)(Z<2/3))/2→B
:PT-On(A,B)
:Goto 1
:End
```

Since your calculators don't work exactly like mine, I'll explain the nuts and bolts behind this program so that you can modify it to fit your machine.

Most of this program is prep work:

- Turn off all functions, so that they don't plot over the fractal
- Set the viewing rectangle to $0 \leq x \leq 1$, $0 \leq y \leq 1$
- Turn off three "corner points"
- Initialize $A = .5$ and $B = 1$ (this turns out not to matter so much)
- Bring up the graphics display

The rest of the program carries out a very simple idea:

- Our current point is (A, B)
- Randomly pick one of the vertices (0,0), (1,0), (.5,1)
- Move halfway from (A, B) to the chosen vertex
- Update the coordinates of (A, B)
- Plot the current point
- Repeat

Syntax notes:

- My `Lbl` might be called `Label` on your calculator.
- The inequality symbols, $<$ and $>$, are found on the `TEST` menu (`2nd-MATH`).
- When I say →, I mean Store. Or `STO`. Or whatever the command is for assigning a value to a variable.

The Greatest Integer Function can be used in place of the `TEST` inequalities if you prefer—but it takes some thought. The random number generator `Rand` gives us a number Z between 0 and 1, so $3Z$ is between 0 and 3. Thus, $[[3Z]]$ is an integer in the interval [0,3); that is, 0, 1, or 2 (with equal probabilities of each). Thus, $[[3Z]]/2$ is one of the numbers 0, .5, 1—the x-coordinates of the three vertices. (On my TI, the GIF is called `Int`, available off of the `Math-Num` menu.) We could replace the line

```
:((A+(Z>1/3)/2+(Z>2/3)/2)/2) → A
```

with

```
:((A+(Int 3Z)/2)/2) → A
```

It's a little trickier to use the GIF instead of the `TEST` inequalities for the y-coordinate. Again, $[[3Z]]$ is one of 0, 1, or 2, but now we have to make sure the correct y-coordinate matches the x-coordinate. $[[3Z]] = 0$ corresponds to the vertex (0,0); $[[3Z]] = 1$ corresponds to (.5,1), and $[[3Z]] = 2$ corresponds to (1,0). That is, the y-coordinate is zero when $[[3Z]]$ is 0 or 2, but one when $[[3Z]] = 1$. If we are desperate to avoid the `TEST` inequalities, we can get it done as follows.

$\text{Int}(3Z)$	0	1	2
$\text{Int}(3Z) - 1$	-1	0	1
$\text{Abs}(\text{Int}(3Z) - 1)$	1	0	1
$1 - \text{Abs}(\text{Int}(3Z) - 1)$	0	1	0

...and these are the y-coordinates, in the correct order. Now we can replace the line

```
:((B+(Z>1/3)(Z<2/3))/2 → B
```

with

```
:((B+1-(Abs (Int (3Z) - 1))/2 → B.
```

I believe we save about 11 bytes of memory by using GIF instead of `TEST`. On my ancient machine, that's around half a percent of the total drive space. On modern calculators, that's utterly insignificant.

Getting away from the technicalities of implementation, there's one big mathematical puzzle here. If we've been taking random steps, why didn't we end up with a noise-like cloud? Why does our graph have any kind of recognizable structure? The answer, which I won't give you now, belongs to transformational geometry.

Finally (and I say "finally" because, sadly, I must write a midterm), there is one incredible connection between Sierpinski's Triangle and Pascal's Triangle (which, in turn, is connected to basically everything else in the world). Do a little web search on Pascal's Triangle. If you don't immediately find a page that shows you the Sierpinksi-Pascal connection, then try this at home: make a big Pascal's Triangle, maybe 16 rows. Shade the odd numbers. (It is conceptually harder but computationally easier to replace every number in Pascal's triangle with a 1 if odd, or a 0 if even. Just remember that 1+1=0, and you won't need to think about 330+462 later.)