Final Exam

This is the final exam for Math 145, Winter 2009. Please write your name clearly at the top of the exam. You may not use any notes or books, nor any calculating or computing devices. Please write your solutions as clearly as you can and include justification for your assertions. All graphs are undirected graphs without loops or multiple edges unless otherwise stated.

1. Suppose G is a graph with n vertices such that every vertex has degree at least $n/2$. Show that G is connected.

If G were disconnected then the smallest connected component C in G would have $\leq n/2$ vertices. The degree of any vertex in C would then be $\leq n/2 - 1$, contradicting the hypothesis. Hence, G is connected.

2. Which complete graphs are Eulerian?

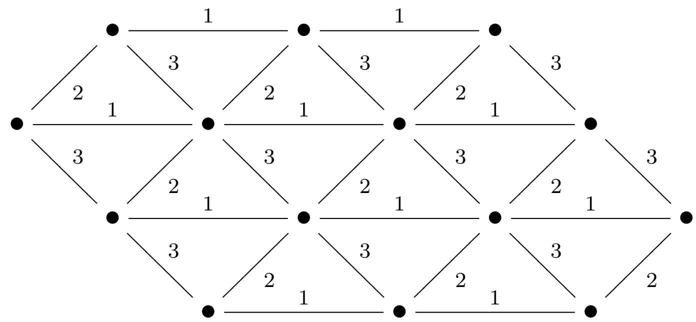
By Euler's Theorem, we have that G is Eulerian if and only if G has no odd degree vertices. Therefore, K_n is Eulerian if and only if n is odd, so that every vertex has degree $n-1$ which is even.

3. Let T be a tree containing a vertex of degree d. Prove that T has at least d leaves.

Let v be a node of T with degree d. Let $W_1, W_2, \ldots, W_d$ be walks that begin from each of the d edges emanating from v and never backtrack along an edge. Since T contains no cycles, these walks cannot intersect themselves, and W_i does not intersect W_j for all $i \neq j$. Otherwise, we would introduce a cycle into the graph. Since the graph is finite, each of these walks must eventually terminate at a vertex with degree 1, so these endpoints of the walks form at least d leaves in the tree.

4. Consider the following graph G shown with edge costs that depend on

the orientation of the edge.



Apply Kruskal's algorithm to find a cheapest spanning tree. Briefly but clearly state how you are running each step of the algorithm.

We choose all of the horizontal 1-edges since they are cheapest, and do not form a cycle. Then, we connect the 4 horizontal paths by 2-edges since they are the next cheapest. There are several ways to do this, but the cost is always

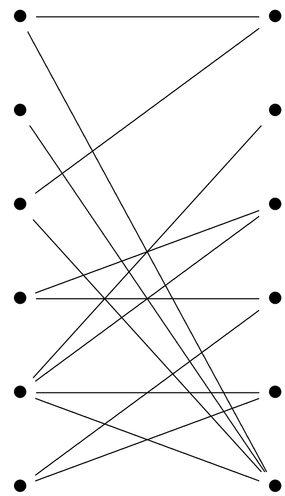
$$(1+1)+(1+1+1)+(1+1+1)+(1+1)+(2)+(2)+(2)=16.$$

5. Let G be a bipartite graph with vertex partition $V = A \sqcup B$ such that $|A| = m = |B|$. Prove that if each node has degree larger than $m/2$ then G has a perfect matching.

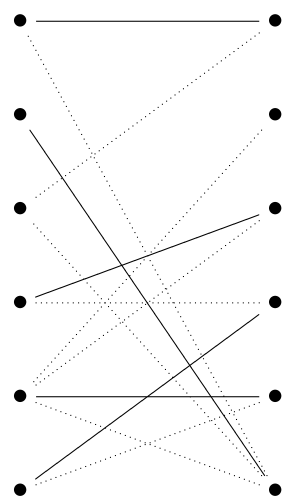
We apply Hall's Theorem. Choose any $S \subset A$ and denote the set of neighbors of S by $N(S)$. We must show that $|N(S)| \geq |S|$. If $|S| \leq m/2$, then this is clear since even a single vertex of S has at least $m/2$ neighbors. So suppose that $|S| > m/2$ but $|N(S)| < |S|$. In this case, all neighbors of the complement $B \setminus N(S)$ must come from $A \setminus S$. But then the nodes of $B \setminus N(S)$ must have degree $\leq |A \setminus S| < m/2$, a contradiction. Therefore, $B \setminus N(S) = \emptyset$, so $|N(S)| = m \geq |S|$, as desired.

6. Find a matching of the following graph G with the maximal number

of matched edges.



We claim that



is a matching with the maximal number of edges. To see this, observe that there can be no augmenting path for this matching because the second vertex on the left side is matched and has no other unmatched edges out. By the result we proved in lecture, this imples that no larger matching exists.

7. Find a list of rankings for hospitals A, B, C and doctors x, y, z, so that no hospital is matched with their first choice in the stable matching obtained from the Gale–Shapley Algorithm.

To avoid having any hospital get their first choice, we can make up a series of moves that we hope will be realizable in the algorithm and then try to construct rankings accordingly. Here is one sequence of events that would work:

A proposes to x, B proposes to x who accepts (so x prefers B to A), rejecting A. A proposes to y who accepts. C proposes to y who rejects (so y prefers A to C), so C proposes to x who accepts (so x prefers C to B), rejecting B. Finally, B proposes to z who accepts. We obtain the following parings.

$$(A, y), (B, z), (C, x).$$

This sequence is implemented by the following preferences.

	#1	#2	#3
A	x	y	z
B	x	z	y
C	y	x	z
x	C	B	A
y	B	A	C
z	A	B	C

8. Describe a way to draw 10 points in the plane in general position that do not contain a convex 6-gon.

Draw two concentric regular 5-cycles as shown in Figure 1.

We call vertices the outer 5-cycle the *outside points*, and the vertices of the inner 5-cycle the *inside points*. We claim that no 6 points span a hexagon. To see this, we just need to check that all subsets of size 6 have a convex hull with less than 6 vertices. This is a finite check that can be expedited by breaking into cases and using the symmetry of the figure.

For example, any 6 points must use some number of outside points. If we use 5 outside points, then all of the inside points are in their convex hull so we can't add a vertex. If we use 4 outside points, then by symmetry it doesn't matter which vertex we omit. There will only be 1 inside point that is not in the convex hull, so we can't obtain 6 vertices. If we use 3 outside points that are consecutive in the outer 5-cycle, then any set of 3 inner points will give a convex hull with only 5 vertices. If we use 3 outside points that are not consecutive in the outer 5-cycle, then there will only be 2 inner points not

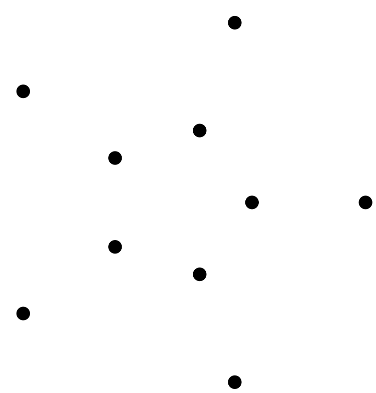
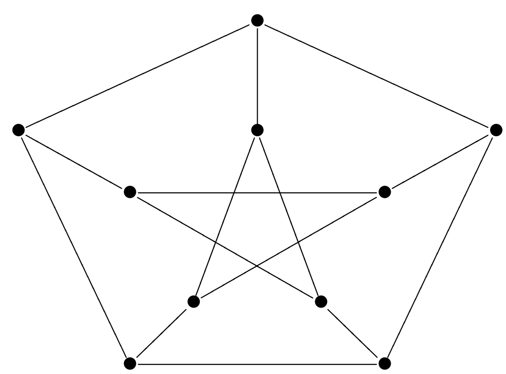


Figure 1: 10 points with no convex 6-gon.

already in the convex hull. If we use 2 outside points that are consecutive in the outer 5-cycle, then any set of 4 inner points will give a convex hull with less than 6 vertices. If we use 2 outside points that are not consecutive in the outer 5-cycle, then any set of 4 inner points will give a convex hull with less than 6 vertices. Finally, if we use only 1 outside point, then there always exists an inside point that is in the convex hull of the chosen outside point and the other inside points.

Hence, there is no convex hexagon among these 10 points.

9. Show that the Peterson graph (shown below) is not planar *using Euler's formula.*



If the Peterson graph were planar, then Euler's formula gives $2 = f - 15 + 10$, so the number of faces $f = 7$. We claim that the smallest cycle in the Peterson graph has length 5. This is a finite check that can be proved using cases in which we consider all ways that we could have a 4-cycle or 3-cycle as a combination of inside points and outside points.

Therefore, each face has at least 5 boundary edges, each counted twice. Thus, the number of edges is $e \geq 35/2 > 15 = e$, a contradiction.

10. Suppose $n \geq 3$. Let G_n be the graph arising from K_n by omitting the edges of a Hamiltonian cycle. Find $\chi(G_n)$.

We omit the outermost cycle in the usual drawing of K_n. Greedily color each vertex as we go around the (omitted) outside cycle. Then the colors are assigned in pairs: $1, 1, 2, 2, 3, 3, \cdots$, so we use $\lceil \frac{n}{2} \rceil$ colors in total.

Now, suppose for the sake of contradiction that there exists coloring of G_n that uses fewer colors. Then some color must be assigned to at least 3 vertices by the Pigeonhole Principle. Observe that in any graph coloring, the vertices of a particular color form a complete graph in the complement graph by definition. But in our case, this would imply that the complement of G_n contains a triangle, which is a contradiction for $n > 3$ since the complement of G_n is the n-cycle.

Hence, the greedy coloring we described is optimal, and $\chi(G_n) = \lceil \frac{n}{2} \rceil$ for $n > 3$, with $\chi(G_3) = 1$.