

## Homework 2

0. Problems from section 2.5 and 3.8:

2.5.4, 2.5.8, 3.8.6.

1. Prove that if we choose 55 distinct numbers from  $\{1, 2, \dots, 100\}$ , there are always two which differ by 10. Also, show that there are always 2 numbers which differ by 12. On the other hand, show that it is possible to choose the 55 numbers so that no two numbers differ by 11.

2. Suppose we color a  $3 \times 7$  grid with two colors. Show that there exists a rectangle consisting of at least 4 squares, such that the corners all have the same color.

For example,

<b>red</b>	<i>blue</i>	<i>blue</i>	<b>red</b>	<i>blue</i>	<i>blue</i>	<i>red</i>
<i>blue</i>	<i>blue</i>	<i>red</i>	<i>red</i>	<i>red</i>	<i>blue</i>	<i>red</i>
<b>red</b>	<i>red</i>	<i>blue</i>	<b>red</b>	<i>blue</i>	<i>red</i>	<i>blue</i>

has such a rectangle given in boldface.

(For a hint, consider breaking the problem into two cases: Either some column is monochromatic (all three entries have the same color), or no column is monochromatic.)

3. Find and prove a formula for  $f(n) = \sum_{i=1}^n i^3$ .

4. Show that any permutation of  $[10]$  either contains an increasing subsequence of length 4, or a decreasing subsequence of length 4.

For example,  $[9, \mathbf{1}, 3, \mathbf{2}, 4, \mathbf{5}, \mathbf{7}, 6, 10, 8]$  has an increasing subsequence of length 4 given in boldface.

(Hint: Label each entry  $p_i$  of the permutation by a pair  $(a_i, b_i)$  where  $a_i$  is the length of the longest decreasing subsequence ending at  $p_i$ , and  $b_i$  is the length of the longest increasing subsequence ending at  $p_i$ . For the example permutation above, these pairs would be

$(1, 1), (2, 1), (2, 2), (3, 2), (2, 3), (2, 4), (2, 5), (3, 5), (1, 6), (2, 6)$ .

Then apply the pigeonhole principle.)