Homework 2

0. Problems from section 2.5 and 3.8:

2.5.4. (a) As a base case, observe that $(3)^2 - 1$ is 8 which is indeed a multiple of 4. Next, assume that $(2(n-1)+1)^2 - 1$ can be written as $4k$. Then,

$$(2n+1)^2 - 1 = (2n)^2 + 2(2n) + 1 - 1 = (2n)^2 - 2(2n) - 1 + 4(2n) + 1$$

$$= (2n-1)^2 + 1 + 4(2n) = 4k + 4(2n) = 4(k+2n)$$

so we have proved the induction hypothesis. Hence, the result is true for all $n \geq 2$. Part (b) is similar.

(Actually, the first equality above already shows that the result is a multiple of 4 without induction, but we have given the inductive argument for completeness.)

2.5.8. (a) There are only 4 colors, so if we remove 5 socks then we must have two socks of the same color by the pigeonhole principle.

(b) There are more black socks than any other color, and in the worst case we might have to remove all 12 black socks before we see another color. Hence, removing 13 socks ensures that we will have at least two colors.

3.8.6. As we walk along the 10×20 grid from $(1,1)$ to $(10,20)$, we can traverse one East-West block (denoted E) or one North-South block (denoted S) at each step. There are 9 total S blocks to traverse, and 19 total E blocks to traverse, so no path can be shorter than $9 + 19 = 28$ blocks. This bound can be attained, for example by walking all of the S blocks followed by all of the E blocks.

In fact, there are many ways to do this. We encode a path as a word of length 28 on the letters E and S such that there are 9 S's and 19 E's. For example, the path above would be encoded

$$SSSSSSSSSEEEEEEEEEEEEEEEEEEE.$$

We count these as $\binom{28}{9}$ because once we choose the positions for the 9 S's, we can fill in the rest of the positions with E's. Of course, this is also the same as $\binom{28}{19}$.

1. Prove that if we choose 55 distinct numbers from $\{1, 2, \ldots, 100\}$, there are always two which differ by 10. Also, show that there are always 2 numbers which differ by 12. On the other hand, show that it is possible to choose the 55 numbers so that no two numbers differ by 11.

Arrange the numbers from [100] in rows of 10:

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Our subset contained 55 entries, and we view these as being distributed into the 10 columns. By the pigeonhole principle, there must be some column with at least 6 entries from our subset. In any fixed column with at least 6 selected entries, we must have that two of the entries lie on adjacent rows, and these adjacent entries differ by 10.

A similar argument works if we arrange the numbers from [100] into 8 rows of 12 with an extra row of 4 at the end.

However, if we arrange the entries of [100] into 9 rows of 11, we *can* select 55 elements such that no two lie in adjacent rows of any column: Choose every other entry (5 entries total) in each of 11 columns. By construction, none of these elements differ by 11.

2. Suppose we color a 3×7 grid with two colors. Show that there exists a rectangle consisting of at least 4 squares, such that the corners all have the same color.

For example,

red	*blue*	*blue*	**red**	*blue*	*blue*	*red*
blue	*blue*	*red*	*red*	*red*	*blue*	*red*
red	*red*	*blue*	**red**	*blue*	*red*	*blue*

has such a rectangle given in boldface.

(For a hint, consider breaking the problem into two cases: Either some column is monochromatic (all three entries have the same color), or no column is monochromatic.)

First observe that if any pair of columns has exactly the same colors, then by the pigeonhole principle at least two of the three entries in the columns have the same color. Hence, we obtain a rectangle with all corners having the same color whenever there are two columns with the same coloring.

As in the hint, we break the problem into two cases. If any column is monochromatic, then we can assume the monochromatic column is

$$\begin{matrix} red \\ red \\ red \end{matrix}$$

(for if it is not then we can switch the colors). If there is any other column with two red entries, then we will have a rectangle with all red corners. But there are only 4 types of columns that do not include two red entries. They are:

$$\begin{matrix} blue \\ blue \\ blue \end{matrix}, \begin{matrix} red \\ blue \\ blue \end{matrix}, \begin{matrix} blue \\ red \\ blue \end{matrix}, \begin{matrix} blue \\ blue \\ red \end{matrix}.$$

If we assign the remaining 6 columns to these four types, we have by the pigeonhole principle that two of the columns have the same type. Hence, we obtain a rectangle with all corners having the same color.

Next, suppose that no column is monochromatic. Then, there are 6 possible colorings for any column: $2^3 = 8$ total colorings minus the two monochromatic colorings. However, if we assign the 7 columns to these 6 types, we have by the pigeonhole principle that two of the columns have the same type. Hence, we obtain a rectangle with all corners having the same color.

3. Find and prove a formula for $f(n) = \sum_{i=1}^{n} i^3$.

If we compute some examples, we find

$$1^3 = 1$$

$$1^3 + 2^3 = 9$$

$$1^3 + 2^3 + 3^3 = 36$$

and

$$1^3 + 2^3 + 3^3 + 4^3 = 100.$$

These are all sqaures, and we might conjecture that $f(n) = \binom{n+1}{2}^2$.

For a proof by induction, our data provides a base case. Suppose that $f(n-1) = \binom{n}{2}^2$. Then,

$$f(n) = f(n-1) + n^3 = \binom{n}{2}^2 + n^3 = \left(\frac{n(n-1)}{2}\right)^2 + n^3$$

$$= \frac{n^4 - 2n^3 + n^2}{4} + \frac{4n^3}{4} = \frac{n^4 + 2n^3 + n^2}{4} = \left(\frac{n(n+1)}{2}\right)^2 = \binom{n+1}{2}^2$$

so we have proved the induction hypothesis. Hence, the result is true for all $n \geq 2$.

4. Show that any permutation of [10] either contains an increasing subsequence of length 4, or a decreasing subsequence of length 4.

For example, $[9, \mathbf{1}, 3, \mathbf{2}, 4, \mathbf{5}, \mathbf{7}, 6, 10, 8]$ has an increasing subsequence of size 4 given in boldface.

As in the hint, we label the ith entry p_i of the permutation by a pair (a_i, b_i) where a_i is the length of the longest decreasing subsequence ending at p_i, and b_i is the length of the longest increasing subsequence ending at p_i. For the example permutation above, these pairs would be

$$(1,1), (2,1), (2,2), (3,2), (2,3), (2,4), (2,5), (3,5), (1,6), (2,6).$$

If there were no increasing subsequence of length 4 and no decreasing subsequence of size 4, then we would have $1 \leq a_i \leq 3$ and $1 \leq b_i \leq 3$ for all i. Hence, there are $3^2 = 9$ different labels that we can assign to the entries of the permutation. Since our permutation has 10 entries, we have by the pigeonhole principle that there are two entries of the permutation with the same label. This will lead to a contradiction:

Suppose the entries are i and j with $i < j$. Then we either have $p_i > p_j$ or $p_i < p_j$. In the first case, we can append p_j to a longest decreasing subsequence that ends at p_i, but this would give $a_j \geq a_i + 1$, contradicting that p_i and p_j have the same label. Similarly, if $p_i < p_j$ we can append p_j to a longest increasing subsequence that ends at p_i, but this would give $b_j \geq b_i + 1$, contradicting that p_i and p_j have the same label.

Since we reach a contradiction in all cases, the hypothesis that there were no increasing or decreasing subsequences of size 4 must have been false.

This proof can be slightly modified to show that any sequence of length at least $(r-1)(s-1)+1$ either has an increasing subsequence of size r or a decreasing subsequence of size s. This result is known as the Erdös–Szekeres Theorem, first proved by them in 1935.