Homework 3

0. Problems from sections 3 and 4:

3.8.10. To distribute n pennies to k children such that each child gets 5 pennies, we can use the same idea as in Section 3.4. If we start out by giving every child 5 pennies, then we only have to distribute $n - 5k$ pennies to k children (with no restrictions). By Theorem 3.4.2, this is

$$\binom{(n-5k)+k-1}{k-1} = \binom{n-4k-1}{k-1}.$$

4.3.5. Let F_n denote the number of tiling of a $2 \times n$ chessboard. Each tiling of a $2 \times n$ chessboard either has the last domino in a vertical position, or the last two columns consist of two dominos arranged horizontally. In the first case, the remainder of the chessboard is a tiling of a $2 \times (n-1)$ board, while in the second case the remainder of the chessboard is a tiling of a $2 \times (n-2)$ board. Hence, we have $F_n = F_{n-1} + F_{n-2}$. By direct computation, we have $F_1 = 1$ and $F_2 = 2$. Hence, F_n are the Fibonacci numbers.

Sloane's on-line encyclopedia of integer sequences has *many* other interpretations of the Fibonacci numbers; see http://www.research.att.com/~ njas/sequences/A000045.

4.3.13. These two initial conditions are enough to determine the sequence because we can solve the recurrence to find a_1. Since $13 = a_2 = a_1 + 2a_0 = a_1 + 2(4)$, we can solve for $a_1 = 5$. Hence,

$$a_5 = a_4 + 2a_3 = (a_3 + 2a_2) + 2(a_2 + 2a_1) = (a_2 + 2a_1) + 2a_2 + 2a_2 + 4a_1$$

$$= 5a_2 + 6a_1 = 65 + 30 = 95.$$

4.3.7. Let a_n denote the number of subsets of $[n]$ with no three consecutive integers. In any valid subset A of $[n]$, exactly one of the following cases occurs:

(1) $n, (n-1) \in A$.

(2) $n \in A$ but $(n-1) \notin A$.

(3) $n \notin A$.

In case (1), we cannot have $n-2 \in A$, so $A \setminus \{n, n-1\}$ is a valid subset of $[n-3]$. In case (2), we have $(n-1) \notin A$ so $A \setminus \{n\}$ is a valid subset of $[n-2]$. In case (3), we have $n \notin A$ and no other restrictions so $A \setminus \{n\}$ is a valid subset of $[n-1]$. Hence, we have

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}.$$

By direct computation, some initial conditions are $a_0 = 1$, $a_1 = 2$, $a_2 = 4$. Sloane's on-line encyclopedia of integer sequences calls these the Tribonacci numbers; see http://www.research.att.com/˜ njas/sequences/A000073.

1. Find a generating function to determine the number of different ways to make n cents from pennies, nickels, dimes and quarters. How would you determine the number of ways to make a dollar out of these coins?

The generating function for the number of ways to make n cents out of pennies is just the geometric series $1 + x + x^2 + \cdots = \frac{1}{1-x}$. The generating function for the number of ways to make n cents out of nickels is $1 + x^5 + x^{10} + x^{15} + \cdots = \frac{1}{1-x^5}$. Similarly, the number of ways to make n cents out of dimes is encoded by $\frac{1}{1-x^{10}}$ and the number of ways to make n cents out of quarters is encoded by $\frac{1}{1-x^{25}}$. Multiplying these together gives the required generating function

$$\frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}$$

$$= (1+x+x^2+\cdots)(1+x^5+x^{10}+\cdots)(1+x^{10}+x^{20}+\cdots)(1+x^{25}+x^{50}+\cdots).$$

We can multiply out the first few terms, obtaining

$$1 + x + x^2 + x^3 + x^4 + 2x^5 + 2x^6 + 2x^7 + 2x^8 + 2x^9 + 4x^{10} + \cdots$$

where for example, $4x^{10}$ corresponds to the fact that there are 4 ways to make 10 cents: one dime, two nickels, one nickel and five pennies, or ten pennies.

The number of ways to make a dollar out of coins is the coefficient of x^{100} in this power series. Using software, this is straightforward to compute and the answer turns out to be 242 ways.

2. Prove the identity

$$\sum_{k=0}^{n} \binom{n}{k}\binom{k}{m} = \binom{n}{m} 2^{n-m}.$$

To give an algebraic proof, observe that if the right side were summed over m, we would be in a position to apply the binomial theorem. So, we multiply both sides by x^m and sum both sides over all m from 0 to n. This gives a generating function identity

$$\sum_{m=0}^{n}\sum_{k=0}^{n}\binom{n}{k}\binom{k}{m}x^m = \sum_{m=0}^{n}\binom{n}{m}2^{n-m}x^m.$$

If we can prove this, then by comparing coefficients on both sides we will obtain the identity we originally wanted to prove. To prove the generating function identity, we expand $(x+2)^m$ in two different ways:

$$\begin{aligned}
(x+2)^n &= (x+2)^n \\
((1+x)+1)^n &= \sum_{m=0}^{n}\binom{n}{m}2^{n-m}x^m \\
\sum_{k=0}^{n}\binom{n}{k}(1+x)^k &= \sum_{m=0}^{n}\binom{n}{m}2^{n-m}x^m \\
\sum_{k=0}^{n}\binom{n}{k}\left(\sum_{m=0}^{k}\binom{k}{m}x^m\right) &= \sum_{m=0}^{n}\binom{n}{m}2^{n-m}x^m \\
\sum_{k=0}^{n}\binom{n}{k}\left(\sum_{m=0}^{n}\binom{k}{m}x^m\right) &= \sum_{m=0}^{n}\binom{n}{m}2^{n-m}x^m
\end{aligned}$$

and interchanging the order of summation on the left side yields the result.

3. How many distinct terms are there in the expansion of

$$(x_1 + x_2 + \cdots + x_m)^n \quad ?$$

For example, when $m = 2$ there are $n+1$ terms corresponding to $k = 0, 1, \ldots, n$ in the expansion $\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$. How many total terms are there in the expansion? For example, when $m = 2$ we have seen that there are 2^n total terms.

The multinomial theorem tells us that

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{n_1+n_2+\cdots+n_m=n}\binom{n}{n_1, n_2, \ldots n_m}x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}.$$

Hence, we must count the number of ways to write n as a sum $n_1+n_2+\cdots+n_m$ where some of the n_i are allowed to be 0. As in Section 3.4, we can represent these as words of length $n+m-1$ that have exactly $m-1$ separators and n 1-entries. For example, we would represent $2+5+3+0=10$ as the word $11|11111|111|$. Hence, these terms are counted by $\binom{n+m-1}{m-1}$.

On the other hand, if we let $x_1 = x_2 = \cdots = x_m = 1$, then we find that

$$m^n = \sum_{n_1+n_2+\cdots+n_m=n} \binom{n}{n_1, n_2, \ldots n_m}$$

is the total number of terms in the expansion of $(x_1 + x_2 + \cdots + x_m)^n$.

4. Use the generating function method from lecture to find a closed formula for the sequence given by

$$a_n = 3a_{n-1} + 1, \qquad a_0 = 0.$$

Some initial terms of the sequence are $0, 1, 4, 13, 40, 121, 364, \ldots$.

If we represent this sequence by a generating function $A(x) = \sum_{n\geq 0} a_n x^n$, the recurrence gives

$$\begin{aligned} a_n &= 3a_{n-1} + 1 \\ a_n x^n &= 3a_{n-1}x^n + x^n \\ \sum_{n\geq 1} a_n x^n &= 3\sum_{n\geq 1} a_{n-1}x^n + \sum_{n\geq 1} x^n \\ A(x) - a_0 &= 3(xA(x)) + \frac{x}{1-x} \\ (1-3x)A(x) &= \frac{x}{1-x} \end{aligned}$$

so

$$A(x) = \frac{x}{(1-x)(1-3x)}.$$

We apply partial fractions to write this in the form

$$\frac{C}{1-x} + \frac{D}{1-3x}$$

where $C = -1/2$ and $D = 1/2$. Hence, we have

$$A(x) = \frac{1}{2}\left(\frac{1}{1-3x} - \frac{1}{1-x}\right)$$

$$= \frac{1}{2}\left((1 + 3x + 3^2x^2 + 3^3x^3 + \cdots) - (1 + x + x^2 + x^3 + \cdots)\right)$$

so we can read off the coefficient of x^n as

$$a_n = \frac{1}{2}\left(3^n - 1\right).$$