Homework 4

0. Problems from sections 7 and 8:

7.3.9. Let V be the vertex set of G. If G is connected then we are done, so suppose G has a proper connected component H and choose any vertex v of H. We have that there are no edges between v and the vertices of $V \setminus H$ in G. When we view this statement in terms of $\bar{G}$, we find that v is connected by an edge in $\bar{G}$ to *every* vertex of $V \setminus H$. Since the same statement is true for any other vertex u in H, we also have that u and v are connected in $\bar{G}$. Therefore, v is connected to every vertex in $\bar{G}$. Hence, $\bar{G}$ is connected.

7.3.10. Since G is connected, it has a spanning tree. Choose one and call it T, and also choose any leaf v of the tree. Then, $T \setminus \{v\}$ is still a spanning tree for $G \setminus \{v\}$, so $G \setminus \{v\}$ is connected.

8.5.3. We can prove this by induction on the number of edges. Fix a graph G with n vertices and $m = 0$ edges. Then, there are $n = n - m$ connected components, so we have our base case. For the induction step, suppose that we have added m edges and we have at least $n - m$ connected components in G. Consider adding another edge e to G. If e connects two previously disconnected components, then the number of components decreases by 1. Otherwise, it stays the same. Hence, we have at least $n-m-1 = n-(m+1)$ components for our graph with $m + 1$ edges. By induction, the result holds for all m.

1. For each sequence A below, either find a graph whose vertex degrees are given by A, or prove that no such graph exists.

(a) $A = (3, 2, 2, 1, 1)$.

We proved in 7.1 that there are always an even number of odd degree nodes. This degree sequence does not have this property, so no such graph exists.

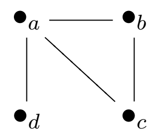
(b) $A = (4, 3, 2, 1, 0)$.

We prove below that there are always two vertices with the same degree. This degree sequence does not have this property, so no such graph exists.

(c) $A = (5, 5, 4, 4, 4)$.

The maximum degree of any vertex in a graph on 5 vertices is 4. Hence, no such graph exists.

(d) $A = (3, 2, 2, 1)$.
A graph with this degree sequence exists. For example,



2. Prove that in any graph, there are two vertices with the same degree.

Let G be a graph, and suppose H is a connected component of G with k vertices. Then the degree of any vertex in H is a number between 1 and $k-1$. But this is only $k-1$ distinct values. To apply the pigeonhole principle, let the k vertices be pigeons and the $k-1$ values be pigeonholes. Then, there must exist two vertices in H with the same degree.

3. Suppose a graph has $m \geq 2$ edges. What is the smallest number of vertices that the graph can have?

Since a graph on n nodes has at most $\binom{n}{2}$ edges, we solve

$$n(n-1) \geq 2m$$

for n in terms of m, obtaining

$$n^2 - n - 2m \geq 0.$$

By the quadratic formula, this implies that

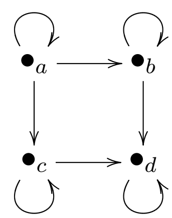
$$n \geq \frac{1 + \sqrt{1 + 8m}}{2}$$

(discarding the other root that gives negative values for n). Since n must be an integer, we actually have

$$n \geq \lceil \frac{1 + \sqrt{1 + 8m}}{2} \rceil.$$

For example, when $m = 1$, the formula gives that $n \geq \lceil \frac{1+\sqrt{1+8}}{2} \rceil = 2$. When $m = \binom{5}{2} = 10$, the formula gives $n \geq \lceil \frac{1+\sqrt{1+80}}{2} \rceil = 5$.

4. Consider the following graph G:



Let f_n denote the number of directed walks of length n in G. Use the adjacency matrix method from lecture to find a recurrence for f_n.

For example, the number of walks of length 1 is the number of edges, so $f_1 = 8$. The number of walks of length 2 turns out to be $f_2 = 14$:

$$aaa, aab, aac, abb, abd, acc, acd, bbb, bbd, bdd, ccc, ccd, cdd, ddd.$$

(Hint: Write down the adjacency matrix A for G and then find the determinant of $I - xA$.)

We write down the adjacency matrix A for G and let $B = I - xA$. Then,

$$B^{-1} = (I - xA)^{-1} = I + xA + x^2A^2 + \cdots$$

is a matrix whose ijth entry is the generating function for the number of walks of length n from node i to node j. Using a formula from linear algebra, we can also write the ijth entry of B^{-1} as the rational function

$$(-1)^{i+j}\frac{\det(B; j, i)}{\det(B)}$$

where $\det(B; j, i)$ means take the determinant after removing the jth row and ith column of B. By a property of rational generating functions that we mentioned in lecture, the denominator $\det(B)$ encodes a recurrence for each entry. Since f_n is the sum of all the entries, $\det(B)$ encodes a recurrence for the sequence f_n.

In our case, we have

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(where the vertices are ordered a, b, c, d). Hence,

$$\det(I - xA) = \det \begin{bmatrix} 1-x & x & x & 0 \\ 0 & 1-x & 0 & x \\ 0 & 0 & 1-x & x \\ 0 & 0 & 0 & 1-x \end{bmatrix} = (1-x)^4 = 1 - 4x + 6x^2 - 4x^3 + x^4.$$

Translating this to a recurrence, we obtain

$$f_n = 4f_{n-1} - 6f_{n-2} + 4f_{n-3} - f_{n-4}.$$

We can generate data to test this by taking matrix powers of A (using software) since f_n is the sum of the entries in the matrix A^n. For example, the sum of the entries in A is 8 so $f_1 = 8$. The sum of the entries in A^2 is $f_2 = 14$. Similarly, we have $f_3 = 22$, $f_4 = 32$, and $f_5 = 44$. This agrees with the recurrence that we found since

$$44 = 4(32) - 6(22) + 4(14) - 8.$$

In principle, you could also use the generating function method that we applied to the Fibonacci numbers in order to solve the recurrence and get an explicit formula for our f_n.