Homework 6

0. Problems from sections 10, 13:

10.4.5, 10.4.15, 13.4.2, 13.4.7.

10.4.5. There is no such bipartite graph. If there were, we would have that the sum of the degrees of the vertices on the left side would be equal to the sum of the degrees on the right side, because each edge has endpoints precisely in these two sets. Since the sum of the given degrees is 44, this means that we must partition the vertices so that the sum of the verticies on the left side is 22. Among the degrees of the vertices on the left side, we must use some number of the 9 copies of 3 given, and if we use 8 copies of 3, then we exceed 22. If we use 7 copies of 3, we still need 1 to add up to 22, and this is not attainable by summing the remaining degrees given. Similarly, if we use 6, 5, 4, 3, or 2 copies of 3, we still need 4, 7, 10, 13, and 16, respectively, none of which are attainable as sums of the remaining degrees $6, 6, 5$. If we use 1 copy of 3, we still need 19 to make 22 which exceeds $17 = 6 + 6 + 5$. Hence, there is no such bipartite graph.

10.4.15. There is a bijection of perfect matchings of the ladder graph to tilings of a $2 \times n$ grid with dominos given by identifying nodes in the graph with cells in the $2 \times n$ grid, and then identifying each vertical edge with a vertical domino, and each horizontal edge with a horizontal domino. Hence, the number of perfect matchings of the $2n$ ladder graph is the nth Fibonacci number: $1, 2, 3, 5, 8, 13, \ldots$.

13.4.2. First note that if $d = 1$, then no graph can satisfy the hypotheses of the problem, because a vertex with degree strictly less than d would imply that the graph is disconnected. Hence, we assume that $d \geq 2$. We prove the statement by induction on the number of vertices. As a base case, observe that when the number of vertices $n = 2$, there is only one connected graph, the path. No matter what degree $d \geq 2$ we pick, we do indeed satisfy the result. Namely, the graph does have a vertex with degree strictly less than d and the graph is d-colorable.

Now, let G be a connected graph such that all vertices have degree at most d and let v be the vertex with degree stictly less than d. Consider the graph G' obtained from G by removing v and all its edges from G. Since G was connected, we have that v was connected to at least one other vertex v', and so v' will now have degree strictly less than d in G'. By induction, we may assume that G' has a d-coloring. We can extend this to a d-coloring of G because v has degree strictly less than d, so we can choose an unused color from $\{1, 2, \ldots, d\}$ for v.

13.4.7. If every face of a planar map has an even number of edges, then every cycle has even length. Therefore, the graph is 2-colorable by Theorem 13.2.1. Hence, the graph is bipartite since we can take all of the vertices colored 1 as our set A, and all of the vertices colored 2 as our set B. The sets A and B partition the vertex set of the graph so that there are no edges within A nor B.

1. Run the Gale–Shapley algorithm on the following rankings of hospitals A, B, C, and doctors x, y, z, to find a stable matching.

	#1	#2	#3
A	x	y	z
B	x	y	z
C	x	y	z
x	A	B	C
y	A	B	C
z	A	B	C

The soap opera goes as follows: A proposes to x who accepts. Then, B proposes to x who declines, so B proposes to y who accepts. Then, C proposes to x who declines, so C proposes to y who declines, so C proposes to z who accepts. Hence, we pair A with x, B with y and C with z.

Notice that the order in which we issue the proposals does not affect the output of the algorithm.

2. The *girth* of a graph G is the minimum length of a cycle contained as a subgraph of G. Prove that every connected planar graph with n vertices, e edges and girth g satisfies

$$e \leq \frac{g}{g-2}(n-2).$$

As in the proof of Theorem 12.2.2 from the book, we can apply Euler's formula

$$n + f = e + 2$$

where f is the number of faces in G. This time, when we count the number edges face by face, we have that each face has at least g edges on its boundary, because the girth of the graph is g. Hence, there are gf total edges, but these are each counted twice because each boundary edge is on the boundary of two distinct faces. Hence, we get $e \geq \frac{gf}{2}$. Solving for f gives

$$f \leq \frac{2e}{g}$$

and substituting into Euler's formula gives

$$e + 2 = n + f \leq n + \frac{2e}{g}$$

so

$$e(g-2) \leq g(n-2)$$

and

$$e \leq \frac{g}{g-2}(n-2)$$

as desired.

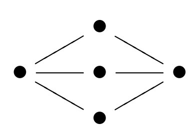
3. Prove that the complete bipartite graph $K_{3,3}$ is not planar, but that the complete bipartite graph $K_{2,3}$ is planar. (Do not merely cite Kuratowski's theorem.)

Suppose for the sake of contradiction that $K_{3,3}$ is planar. The smallest cycle in any graph has size 3, but $K_{3,3}$ cannot contain a triangle because the triangle is not bipartite. Hence, $K_{3,3}$ has girth g at least 4. By the previous problem, the number of edges e in $K_{3,3}$ satisfies

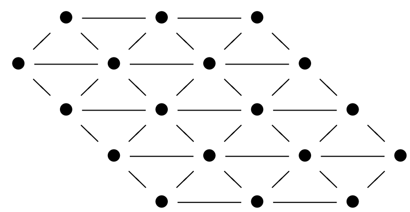
$$e \leq \frac{g}{g-2}(n-2) \leq \frac{4}{4-2}(n-2) = 2(n-2) = 8$$

where $n = 6$ is the number of vertices. But this contradicts that $e = 9$. Hence, $K_{3,3}$ is not planar.

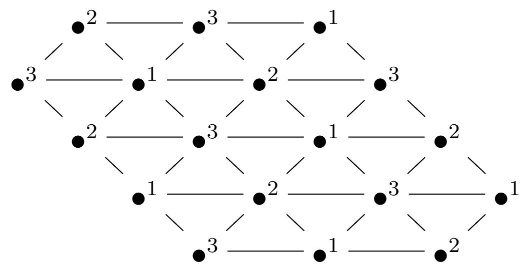
To see that $K_{2,3}$ is planar, we can draw it as a planar map:



4. What is the chromatic number of the following graph G?



The graph contains triangles as subgraphs, so the chromatic number is at least 3. In fact, we can color G with three colors as shown below. Hence, $\chi(G) = 3$.



5. Let T be a tree. Describe an algorithm to produce a 2-coloring of T.

Choose a node v that we will designate as the root vertex. Then every other node in T has a unique path to v, because if there were two distinct paths then we would obtain a cycle contradicting that T is a tree. Therefore, we can assign a nonnegative integer $d(x)$ to each node x in T that is the length of the unique path from v to x. We then color all of the vertices with even $d(x)$ red, and all of the vertices with odd $d(x)$ blue.

Now, whenever there are two vertices x and y connected by an edge, we either have that x is closer to the root than y or vice versa. Hence, $d(x) = d(y) \pm 1$, so the vertices x and y have distinct colors. Therefore, our assignment of colors is a valid 2-coloring of T.

6. Show that every graph G has a vertex ordering for which the greedy algorithm uses only $\chi(G)$ colors. On the other hand, find a bipartite graph on $2n$ vertices ordered in such a way that the greedy algorithm uses n colors (rather than $\chi = 2$ colors).

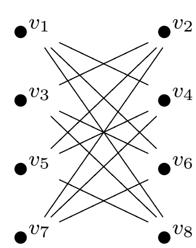
Fix a coloring of the vertices of G that uses $c = \chi(G)$ colors. Let $v_{i,1}, v_{i,2}, \ldots, v_{i,k_i}$ denote be all of the vertices that are colored i, where $1 \leq i \leq c$. Now we order the vertices so that all of vertices colored 1 come first, followed by all the vertices colored 2, and so on, finishing with all of the vertices colored c. Within each set of vertices that all have the same color, any ordering of the vertices may be used. Hence, our vertex ordering is

$$v_{1,1}, v_{1,2}, \ldots, v_{1,k_1}, v_{2,1}, v_{2,2}, \ldots, v_{2,k_2}, \ldots, v_{c,1}, v_{c,2}, \ldots, v_{c,k_c}.$$

When we run the greedy algorithm with this ordering of the vertices, we color all of the vertices $v_{1,j}$ $(1 \leq j \leq k_1)$ with color 1, since no pair of vertices in this set can be connected by an edge (because they all had the same color in the fixed coloring we started with). Similarly, we color all the vertices $v_{i,j}$ $(1 \leq j \leq k_i)$ with one of the colors $\{1, 2, \ldots, i\}$, because no pair of vertices in this set can be connected by an edge (because they had the same color in the fixed coloring we started with). This means that every vertex $v_{i,j}$ is connected to vertices with colors from the set $\{1, 2, \ldots, i-1\}$, and only if $v_{i,j}$ is connected to vertices with all of these colors do we need to use color i.

Hence, the greedy algorithm uses at most c colors, and since c is the minimal number of colors possible, we have that the greedy algorithm uses exactly c colors.

On the other hand, we can construct a bipartite graph and an ordering of its vertices for which the greedy algorithm fails badly. Consider the following graph.



$\vdots$

We say that the vertices v_{2i-1} and v_{2i} lie on *level* i of the graph. This graph is constructed by connecting each vertex v_i on the left side to all of the vertices on the right side that are stictly above the level of v_i, and similarly connecting each vertex v_j on the right side to all of the vertices on the left side that are strictly above the level of v_j.

If we order the vertices $v_1, v_2, v_3, \ldots$ as shown, then the vertices on the first level will receive color 1 in the greedy algorithm, since they are not connected. The vertices v_3, v_4 on level 2 will receive color 2 in the greedy algorithm, since they are both connected to the vertices on level 1, but not to each other. Similarly, the vertices on level i will receive color i, because they are connected to all of the previous levels by construction. Hence, the greedy algorithm will produce an n-coloring of this graph, when in fact a 2-coloring is possbile.