Midterm Exam

Choose any four of the following problems.

1. Let $F_1 = 1$, $F_2 = 1$ and F_n be the nth Fibonacci number. Show that if n is a multiple of 4, then F_n is a multiple of 3. For example, $F_4 = 3$ and $F_8 = 21$.

Since $F_3 = 1 + 1$ and $F_4 = 2 + 1$, we have $F_4 = 3$ as a base case. For the induction step, suppose that F_{n-4} is a multiple of 3. Then,

$$F_n = F_{n-1} + F_{n-2} = (F_{n-2} + F_{n-3}) + (F_{n-3} + F_{n-4})$$

$$= ((F_{n-3} + F_{n-4}) + F_{n-3}) + (F_{n-3} + F_{n-4}) = 3F_{n-3} + 2F_{n-4}.$$

Hence, F_n is a multiple of 3. Therefore, F_{4k} is a multiple of 3 for all $k \geq 1$ by induction.

2. Let $n \geq 3$. Find the number of permutations of $[n]$ in which none of the first three integers are in their natural position. For example, when $n = 3$ there are two such permutations

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{matrix} \quad \text{and} \quad \begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{matrix}.$$

We use the inclusion-exclusion formula. Let A_i be the set of permutations of $[n]$ with the integer i in its natural position. Then, the permutations that do not lie in any A_i for $i \in \{1, 2, 3\}$ are:

$$n! - \sum_{J \subset [3] \text{ with } |J|=1} \bigcap_{i \in J} |A_i| + \sum_{J \subset [3] \text{ with } |J|=2} \bigcap_{i \in J} |A_i| - \sum_{J \subset [3] \text{ with } |J|=3} \bigcap_{i \in J} |A_i|$$

$$= n! - \binom{3}{1}(n-1)! + \binom{3}{2}(n-2)! - \binom{3}{3}(n-3)!.$$

$$= n! - 3(n-1)! + 3(n-2)! - (n-3)!.$$

3. Let $A(x) = \frac{1}{1-x}$. What sequence does the generating function $A(x)$ encode? What sequence is encoded by the generating function

$$xA'(x) = x\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{x}{(1-x)^2}?$$

The expansion of $A(x) = \frac{1}{1-x}$ as a power series is the geometric series $1+x+x^2+x^3+\cdots$. Reading the coefficients, we see that the generating function $A(x)$ encodes the sequence $a_0 = 1 = a_1 = a_2 = \cdots$. If we differentiate $A(x)$ term by term as a power series, we get

$$x\frac{d}{dx}\left[1+x+x^2+x^3+\cdots\right] = x\left(0+1+2x+3x^2+4x^3+\cdots\right)$$

$$= x+2x^2+3x^3+4x^4+\cdots$$

and reading the coefficient of x^n, we see that $A'(x)$ encodes the sequence $a_n = n$.

4. Let $n \geq 3$ and suppose $x_1, x_2, \ldots, x_n$ are n points in the plane with no three points on the same line. Imagine coloring these n points with k colors. How large can k be (in terms of n) so that no matter how we color the points, there always exists a triangle with all three vertices having the same color?

For example, $k = 1$ certainly guarantees that there exists a monochromatic triangle, but if $k = n$ then there exists a coloring without a monochromatic triangle, so the optimal value of k lies between 1 and n.

We view the colors as boxes and the vertices as pigeons to apply the Pigeonhole Principle. Then, we are guaranteed a monochromatic triangle if and only if $n \geq 2k+1$. (Indeed, if there is no monochromatic triangle, then every color has at most 2 vertices so n would be $\leq 2k$.) This inequality is equivalent to

$$k \leq \frac{n-1}{2}$$

and since k must be an integer, we actually have

$$k \leq \lfloor\frac{n-1}{2}\rfloor.$$

For example, when n is 3 or 4, $k = 1$ is optimal, but for $n = 5$, any 2-coloring of the points must contain a monochromatic triangle.

5. In how many ways can we seat 8 people in a line if Person 1 refuses to sit next to Person 2?

We count the forbidden seatings. We must avoid seating 1 next to 2 in each of the 7 pairs of adjacent positions on the line. In each case, we must avoid the seating orders 12 and 21 for the adjacent positions, but the remaining 6 positions can be filled without restriction. Hence, there are $2{\cdot}7{\cdot}6!$ forbidden seatings. Therefore, there are

$$8! - 2 \cdot 7 \cdot 6! = 6!(56 - 14) = 42 \cdot 6!$$

valid seatings.