Final Exam

This is the final exam for Math 146, Spring 2008. The exam has 100 points, and you have 120 minutes to complete this exam. You may not use any notes or books, nor any calculating or computing devices. Please give *as much justification as you can* for all of your solutions.

1. (15 points) Let a_n ($n \geq 0$) be a fixed sequence with ordinary generating function $A(x) = \sum_{n\geq 0} a_n x^n$ and let b_n be another sequence defined by

$$b_n = (2^n+1)a_0 + (2^{n-1}+1)a_1 + (2^{n-2}+1)a_2 + \cdots + (2^1+1)a_{n-1} + (2)a_n$$

for all $n \geq 0$. Find the ordinary generating function $B(x) = \sum_{n\geq 0} b_n x^n$ in terms of $A(x)$.

Applying the product rule (3) to the recurrence gives that $B(x)$ is the Cauchy product of the ordinary generating functions $A(x)$ and

$$\sum_{n\geq 0}(2^n+1)x^n = \sum_{n\geq 0}(2x)^n + \sum_{n\geq 0} x^n = \frac{1}{1-2x} + \frac{1}{1-x} = \frac{2-3x}{(1-2x)(1-x)}.$$

Hence, $B(x) = A(x)\frac{2-3x}{(1-2x)(1-x)}$.

2. (15 points) Let a_n be the number of nonnegative integers strictly less than 10,000 whose digits sum to n. For example, $a_2 = 10$ corresponding to the numbers

$$2000, 200, 20, 2, 1100, 1010, 1001, 110, 101, 11.$$

Find the ordinary generating function for $\{a_n\}_{n\geq 0}$.

(Hint: Every nonnegative integer less than 10,000 can be encoded as a 4-digit number if we allow leading zeros. Use this encoding together with one of the ogf rules to solve the problem.)

Let d_1, d_2, d_3, d_4 be the digits of such an integer (including leading zeros), so each d_i is an integer between 0 and 9. For example, the number 20 would be encoded as 0020, where $d_1 = 0$, $d_2 = 0$, $d_3 = 2$ and $d_4 = 0$. Then by the power rule (4) we have

$$\{a_n\}_{n\geq 0} = \{\sum_{d_1+d_2+d_3+d_4=n} 1\}_{n\geq 0} = \{\sum_{d_1+d_2+d_3+d_4=n} b_{d_1}b_{d_2}b_{d_3}b_{d_4}\}_{n\geq 0}$$

$$\stackrel{ogf}{\leftrightarrow} (\sum_{i\geq 0}^{9} b_i x^i)^4$$

where $b_i = 1$ for $0 \le i \le 9$ and $b_i = 0$ otherwise. Hence, we obtain

$$A(x) = \sum_{n\ge 0} a_n x^n = (1 + x + x^2 + \cdots + x^8 + x^9)^4 = (\frac{1 - x^{10}}{1 - x})^4.$$

3. (15 points) Let $a_n = \sum_{i=1}^{n} i$ be the sum of the first n positive integers. For example, $a_0 = 0$, $a_1 = 1$, $a_2 = 1 + 2 = 3$ and $a_3 = 1+2+3 = 6$. Find a closed form expression for the ordinary generating function of a_n.

Using the geometric series and applying the partial sum rule (5) three times, we obtain

$$\frac{x}{1-x} \overset{ogf}{\leftrightarrow} \{1\}_{i\ge 1}$$
$$\frac{x}{(1-x)^2} \overset{ogf}{\leftrightarrow} \{i\}_{i\ge 1}$$
$$\frac{x}{(1-x)^3} \overset{ogf}{\leftrightarrow} \{\sum_{i=1}^{n} i\}_{n\ge 1}$$

so

$$A(x) = \sum_{n\ge 0} a_n x^n = \frac{x}{(1-x)^3}.$$

4. (20 points) Let a_n be the number of permutations p of n such that:

- p has an odd number of cycles, and
- the length of every cycle in p is even.

(a) Show that the exponential generating function of $\{a_n\}_{n\ge 0}$ is $\frac{x^2}{2(1-x^2)^{1/2}}$.
(Hint: To include only even (or odd) terms of a generating function, say $F(x)$, recall that we add (or subtract) $F(-x)$ to cancel out half the terms and then divide by 2. The generating function $\sinh x = \sum_{n\ge 1, n \text{ odd}} \frac{x^n}{n!} = \frac{e^x - e^{-x}}{2}$ is an example of this.)

We apply the exponential formula. The cards are cycles and the hands are permutations. Therefore, the deck enumerator is

$$D(x) = \sum_{n\ge 0, n \text{ even}} (n-1)! \frac{x^n}{n!} = \sum_{n\ge 0, n \text{ even}} \frac{x^n}{n}$$
$$= \frac{1}{2}(\log \frac{1}{1-x} + \log \frac{1}{1-(-x)}) = \log(\frac{1}{(1-x)(1+x)})^{1/2}.$$

Using the restricted exponential, we obtain the hand enumerator

$$H(x) = \sum_{n\geq 0, n \text{ odd}} \frac{1}{n!} D(x)^n = \sinh D(x)$$

$$= \frac{1}{2}(e^{\log(\frac{1}{(1-x^2)})^{1/2}} - e^{-\log(\frac{1}{(1-x^2)})^{1/2}})$$

$$= \frac{1}{2}(\frac{1}{(1-x^2)^{1/2}} - (1-x^2)^{1/2})$$

$$= \frac{1}{2}\frac{1-(1-x^2)}{(1-x^2)^{1/2}} = \frac{1}{2}\frac{x^2}{(1-x^2)^{1/2}}.$$

(b) Find an explicit formula for the numbers a_n.
(Hint: Recall that $\frac{1}{(1-x)^k} = \sum_{n\geq 0}\binom{n+k-1}{n}x^n$.)

We found the *exponential* generating function for a_n in part (a), so

$$a_n = [\frac{x^n}{n!}]\frac{1}{2}\frac{x^2}{(1-x^2)^{1/2}} = \frac{1}{2}n![x^{n-2}]\frac{1}{(1-x^2)^{1/2}}$$

$$= \frac{1}{2}n![x^{n-2}]\sum_{n\geq 0}\binom{n+\frac{1}{2}-1}{n}x^{2n}$$

$$= \frac{1}{2}n!\binom{\frac{n-2}{2}-\frac{1}{2}}{\frac{n-2}{2}} \text{ if } n \text{ is even and } n \geq 2.$$

Otherwise, $a_n = 0$ if n is odd.

5. (15 points) Let a_n be the number of set partitions of $[n]$ such that each class in the set partition has at least 3 elements. For example, $a_6 = 11$ corresponding to the set partitions

$$123456, 123|456, 124|356, 125|346, 126|345, 134|256,$$

$$135|246, 136|245, 145|236, 146|235, 156|234$$

Find a closed form expression for the exponential generating function of a_n.

We apply the exponential formula. The standard cards are sets $[k] = \{1, 2, \ldots, k\}$ that form the classes in the set partition of $[n]$. The hands are set partitions of $[n]$. Each deck D_k contains only the single card $[k]$, so the deck enumerator for this problem is

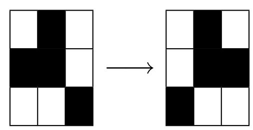
$$D(x) = \sum_{n\geq 3}\frac{x^n}{n!} = e^x - 1 - x - \frac{x^2}{2}.$$

By the exponential formula, the hand enumerator is

$$H(x) = \exp D(x) = \exp\,(e^x - 1 - x - \frac{x^2}{2})$$

and this is the desired generating function.

6. (20 points) We say that a *formal painting* is a coloring of the squares in a 3×3 grid using at most k colors. Two formal paintings are *essentially the same* if one can be obtained from the other by a rotation of the entire painting. For example, the two formal paintings with $k = 2$ colors below are essentially the same because they are related by a 90-degree rotation.



Find the number of essentially different formal paintings with at most k colors.

(a) Begin by writing down the automorphism group G of all rotations acting on the square 3×3 grid

1	2	3
8	9	4
7	6	5

There are 4 rotations on the square. We write the corresponding permutations of the grid using the numbering above:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 5 & 6 & 7 & 8 & 1 & 2 & 9 \end{bmatrix} = (1357)(2468)(9) \leftrightarrow x_4^2 x_1$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 & 9 \end{bmatrix} = (15)(26)(37)(48)(9) \leftrightarrow x_2^4 x_1$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 8 & 1 & 2 & 3 & 4 & 5 & 6 & 9 \end{bmatrix} = (1753)(2864)(9) \leftrightarrow x_4^2 x_1$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{bmatrix} = (1)(2)(3)(4)(5)(6)(7)(8)(9) \leftrightarrow x_1^9$$

(b) Next, compute the cycle index Z_G of G.

Summing the monomials we obtained in (a), we have

$$Z_G(x_1, x_2, \ldots, x_9) = \frac{1}{4}(x_2^4 x_1 + 2x_4^2 x_1 + x_1^9)$$

(c) Apply Pólya's theorem to get a formula for the number of essentially different formal paintings that use at most k colors.

By Pólya's theorem, this is

$$Z_G(k, k, \ldots, k) = \frac{1}{4}(k^5 + 2k^3 + k^9).$$