Midterm Exam

This is the midterm exam for Math 146, Spring 2008. The exam has 100 points, and you have 50 minutes to complete this exam. You may not use any notes or books, nor any calculating or computing devices. Please give *as much justification as you can* for all of your solutions.

1. (30 points) Let

$$a_n = \sum_{i,j\geq 0 \text{ with } i+j=n} \binom{n}{i} \frac{2^i}{3^j}.$$

Find a simple closed form for the exponential generating function $f(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$.

Let $\{2^n\}_{n\geq 0} \overset{egf}{\leftrightarrow} g(x) = e^{2x}$ and $\{(\frac{1}{3})^n\}_{n\geq 0} \overset{egf}{\leftrightarrow} h(x) = e^{\frac{1}{3}x}$. Then,

$$f(x) = \sum_{n\geq 0} (\sum_{k\geq 0}^{n} \binom{n}{k} 2^k (\frac{1}{3})^{n-k}) \frac{x^n}{n!} = g(x)h(x) = e^{\frac{7}{3}x}$$

by the Cauchy product rule.

2. (30 points) A box contains an infinite number of red, green and blue marbles. Give the ordinary generating function for number of ways to choose collections of n marbles from the box such that there is

- no more than 1 red marble,
- at least 6 green marbles, and
- an even number of blue marbles

in each collection. (Hint: Start by writing generating functions for each condition separately and then try to combine them.)

Let $R(x), G(x)$ and $B(x)$ be the generating functions for the number of red, green and blue marbles we choose, respectively, with respect to the number of marbles. Then,

$$R(x) = 1 + x$$

corresponding to the choice of 0 red marbles, or 1 red marble,

$$G(x) = x^6 + x^7 + x^8 + \cdots = \frac{x^6}{1-x}$$

corresponding to the choice of 6 green marbles, 7 green marbles, etc. and

$$B(x) = 1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2}$$

corresponding to the choice of 0 blue marbles, 2 blue marbles, and so on. Now a complete collection of n marbles satisfying the conditions corresponds uniquely to a single term in the product

$$R(x)G(x)B(x) = (1+x)(x^6+x^7+x^8+\cdots)(1+x^2+x^4+\cdots) = \frac{(1+x)x^6}{(1-x)(1-x^2)}.$$

This simplifies to

$$\frac{x^6}{(1-x)^2} = \sum_{n\geq 6}(n-5)x^n$$

yielding an exact formula for the number of such collections.

3. (40 points) Recall that a permutation of n is a bijection $p : \{1,\dots n\} \to \{1,\dots n\}$ that we often write in two-rowed notation as

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ p(1) & p(2) & p(3) & \cdots & p(n) \end{bmatrix}.$$

Let T_n be the set of permutations p of n with $\max|p(i)-i| = 1$. The first few such sets are:

$$T_2 = \{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\}$$

$$T_3 = \{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}\}$$

$$T_4 = \{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{bmatrix}\}$$

Find a simple closed form for the ordinary generating function $f(x) = \sum_{n\geq 0}|T_n|x^n$. (Hint: Start by finding a recurrence for these permutations involving the entry n.)

The condition for a permutation to be in T_n says that each adjacent pair of entries is either $\begin{bmatrix} i & i+1 \\ i+1 & i \end{bmatrix}$ or $\begin{bmatrix} i & i+1 \\ i & i+1 \end{bmatrix}$, and that at least one pair of entries in the permutation must be out of order (since $\max|p(i)-i| \neq 0$).

Hence, each permutation p in T_n either looks like $\begin{bmatrix} n-1 & n \\ n & n-1 \end{bmatrix}$ or $\begin{bmatrix} n-1 & n \\ n-1 & n \end{bmatrix}$ in the last pair of positions. In the first case, we can view p as a permutation in $T_{n-2} \cup \{\begin{bmatrix} 1 & 2 & \cdots & n-2 \\ 1 & 2 & \cdots & n-2 \end{bmatrix}\}$ by removing the last two entries. In the second case, we can view p as a permutation in T_{n-1} by removing the last entry. Hence, we have

$$T_n = T_{n-1} + T_{n-2} + 1$$

which is similar to the Fibonacci recurrence. Note the initial conditions $T_1 = 0$, $T_2 = 1$ so the recurrence gives $T_0 = 0$.

To obtain the generating function, we use the shift rule

$$T_{n+2} - T_{n+1} - T_n = 1$$
$$\overset{ogf}{\leftrightarrow} (f(x) - xT_1 - T_0)/x^2 - (f(x) - T_0)/x - f(x) = \frac{1}{1-x}$$

so

$$(1 - x - x^2)f(x) = \frac{x^2}{1-x}$$

and

$$f(x) = \frac{x^2}{(1-x)(1-x-x^2)}.$$