Homework 1

1.7.1. (a) We are looking for $A(x) = \sum_{n\geq 0} a_n x^n$, which is

$$0 + x + 2x^2 + 3x^3 + \cdots .$$

But we can almost recognize this as the derivative of $\frac{1}{1-x}$ since if we differentiate $1 + x + x^2 + x^3 + x^4 + \cdots$ term by term we obtain

$$D(\frac{1}{1-x}) = 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots .$$

If we then multiply this by x, we get the desired generating function. Hence,

$$A(x) = xD(\frac{1}{1-x}) = \frac{x}{(1-x)^2}.$$

(b) Here α and β are assumed to be constants. Hence, by linearity we obtain

$$A(x) = \sum_{n\geq 0}(\alpha n + \beta)x^n = \alpha \sum_{n\geq 0} nx^n + \beta \sum_{n\geq 0} x^n = \frac{\alpha x}{(1-x)^2} + \frac{\beta}{1-x}.$$

(c) The same trick as in (a) will work here if just apply differentiation and multiplication by x to $\frac{1}{1-x}$ *twice*. We obtain

$$A(x) = \sum_{n\geq 0} n^2 x^n = xDxD\frac{1}{1-x} = xDx(1-x)^{-2} = x((1-x)^{-2} + 2x(1-x)^{-3}).$$

(d) Putting together the previous three parts and using linearity, we obtain

$$A(x) = \sum_{n\geq 0}(\alpha n^2 + \beta n + \gamma)x^n = \alpha \sum_{n\geq 0} n^2 x^n + \beta \sum_{n\geq 0} nx^n + \gamma \sum_{n\geq 0} x^n$$

$$= \alpha(xDxD)\frac{1}{1-x} + \frac{\beta x}{(1-x)^2} + \gamma\frac{1}{1-x}.$$

(f) Here we are looking for $A(x) = \sum_{n\geq 0} 3^n x^n = \sum_{n\geq 0}(3x)^n$. But this is simply a geometric series:

$$A(x) = \frac{1}{1-3x}.$$

1.7.3. For these problems, we are given that $f(x) = \sum_{n\geq 0} a_n x^n$.

(a) Here c is assumed to be a constant. Hence, we are adding

$$c + cx + cx^2 + cx^3 + \cdots = \frac{c}{1-x}$$

to $f(x)$.

(b) Here α is assumed to be a constant. Hence, by linearity we have

$$\sum_{n\geq 0} \alpha a_n x^n = \alpha \sum_{n\geq 0} a_n x^n = \alpha f(x).$$

Hence the ogf of our sequence is $\alpha f(x) + \frac{c}{1-x}$.

(c) This sequence can be obtained by just differentiating and then multiplying by x as in 1.7.1 (a). Hence, we have

$$xDf(x) = xD\sum_{n\geq 0} a_n x^n = x(0 + a_1 + 2a_2 x + 3a_3 x^2 + \cdots)$$

$$= a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots$$

(e) This sequence is obtained by removing the constant term. Hence, we have

$$f(x) - a_0 = f(x) - f(0).$$

(g) This sequence is obtained by removing all the odd terms of the power series. Since $(-x)^{2n} + x^{2n} = 2x^{2n}$ but $(-x)^{2n+1} + x^{2n+1} = 0$, we have

$$\frac{f(-x) + f(x)}{2} = \frac{1}{2}((a_0 - a_1 x + a_2 x^2 - a_3 x^3 + \cdots) + (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots))$$

$$= a_0 + a_2 x^2 + a_4 x^4 + \cdots$$

as desired.