Homework 2

1.7.5. (a) $[x^n]e^{2x}$ is the coefficient of x^n in $\sum_{n\geq 0}\frac{1}{n!}(2x)^n$, which is $\frac{2^n}{n!}$.

(b) $[x^n/n!]e^{\alpha x}$ is the coefficient of $x^n/n!$ in $\sum_{n\geq 0}\frac{1}{n!}(\alpha x)^n$, which is α^n.

(c) Recalling the power series for $\sin x$, we have that $[x^n/n!]\sin x$ is the coefficient of $x^n/n!$ in $\sum_{n\geq 0}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$, which is $(-1)^{(n+1)/2}$ for n odd and 0 otherwise.

(d) Since we know $a \neq b$ we can find $[x^n]\frac{1}{(1-ax)(1-bx)}$ using partial fractions. That is,

$$\frac{1}{(1-ax)(1-bx)} = \frac{A}{(1-ax)} + \frac{B}{(1-bx)}$$

where $A = \frac{1}{(1-ax)(1-bx)}(1-ax)$ with $x = 1/a$ which is $\frac{a}{a-b}$ and by similar reasoning $B = \frac{b}{b-a}$. Hence, the coefficient of x^n is

$$\frac{a^{n+1}-b^{n+1}}{a-b}.$$

(e) Expanding the binomial, we have $(1+x^2)^m = \sum_{k=0}^{m}\binom{m}{k}x^{2k}$ so the coefficient of x^n is $\binom{m}{n/2}$ if n is even and 0 otherwise.

1.7.6. We have done several of these in lecture, so I'll just do (c) here. Applying the method, let $A(x) = \sum_{n\geq 0} a_n x^n$. Then, the recurrence gives

$$\sum_{n\geq 0} a_{n+2}x^n = \sum_{n\geq 0} 2a_{n+1}x^n - \sum_{n\geq 0} a_n x^n$$

which is

$$\frac{A(x)-a_1x-a_0}{x^2} = 2(\frac{A(x)-a_0}{x}) - A(x)$$

yielding

$$A(x)(1-2x+x^2) = x$$

so $A(x) = \frac{x}{1-2x+x^2}$.

1.7.11. Observe that each subset S of $[n]$ counted by $f(n)$ falls into exactly one of the following cases.

1. S contains n, in which case $S - \{n\}$ is a subset of $[n-2]$ counted by $f(n-2)$.

2. S does not contain n, in which case S is a subset of $[n-1]$ counted by $f(n-1)$.

Hence, $f(n) = f(n-1) + f(n-2)$ and $f(0) = 1$ (corresponding to the empty set) while $f(1) = 2$ (corresponding to $S = \{1\}$ and $S = \emptyset$). This is a familiar sequence: $f(n)$ is the $(n+2)$nd Fibonacci number.

1.7.12. Begin by observing that each subset S of $[n]$ of size k counted by $f(n,k)$ falls into exactly one of the following cases.

1. S contains n, in which case $S - \{n\}$ is a subset of $[n-2]$ of size $k-1$, so is counted by $f(n-2, k-1)$.

2. S does not contain n, in which case S is a subset of $[n-1]$ of size k, so is counted by $f(n-1, k)$.

Here, we have $f(n,k) = f(n-1,k) + f(n-2,k-1)$ with initial values $f(0,0) = 1 = f(1,0)$ (corresponding to $\emptyset$), and $f(1,1) = 1$ (corresponding to $\{1\}$). From these we can compute a "triangle of numbers"

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	1				
$n=1$	1	1			
$n=2$	1	2	0		
$n=3$	1	3	1	0	
$n=4$	1	4	3	0	
$n=5$	1	5	6	1	0
$n=6$	1	6	10	4	0

Finding the generating function and explicit formula are described in the back of the book and follow from the same method we have been using in lecture. Note the use of Equation (1.31).