Homework 3

1.7.2. (a) We want $\sum_{n\geq 0} \frac{n}{n!}x^n$ which we can write as $\sum_{n\geq 1} \frac{n}{n!}x^n = x\sum_{n\geq 1} \frac{1}{(n-1)!}x^{n-1}$ which is xe^x. Alternatively, we can use the same trick that we used on the first assignment and apply xD to the exponential generating function e^x that encodes the sequence of all 1's.

(b) Using (a) and linearity, we have $\sum_{n\geq 0} \frac{a_n}{n!}x^n = \alpha xe^x + \beta e^x$.

(c) Applying the result of (a) twice, we obtain $(xD)(xD)e^x = (xD)xe^x = x(e^x + xe^x) = (x + x^2)e^x$.

(f) We want $\sum_{n\geq 0} \frac{3^n}{n!}x^n$ which we can write as $\sum_{n\geq 0} \frac{1}{n!}(3x)^n = e^{3x}$.

(g) Using (f) and linearity, we have $\sum_{n\geq 0} \frac{a_n}{n!}x^n = 5e^{7x} - 3e^{4x}$.

1.7.4. For these problems, we are given that $f(x) = \sum_{n\geq 0} \frac{a_n}{n!}x^n$.

(a) Here c is assumed to be a constant. Hence, we are adding

$$c + cx + \frac{c}{2!}x^2 + \frac{c}{3!}x^3 + \cdots = ce^x$$

to $f(x)$.

(b) Here α is assumed to be a constant. Hence, by linearity we have

$$\sum_{n\geq 0} \frac{\alpha a_n}{n!}x^n = \alpha \sum_{n\geq 0} \frac{a_n}{n!}x^n = \alpha f(x).$$

Hence the egf of our sequence is $\alpha f(x) + ce^x$.

(c) This sequence can be obtained by just differentiating and then multiplying by x as in 1.7.2 (a). Hence, we have

$$xDf(x) = xD\sum_{n\geq 0} \frac{a_n}{n!}x^n = x(0 + a_1 + \frac{2}{2!}a_2 x + \frac{3}{3!}a_3 x^2 + \cdots)$$

$$= a_1 x + \frac{2}{2!}a_2 x^2 + \frac{3}{3!}a_3 x^3 + \cdots$$

(e) This sequence is obtained by removing the constant term. Hence, we have

$$f(x) - \frac{a_0}{0!} = f(x) - f(0).$$

(By the way, $0! = 1$ by definition.)

(g) This sequence is obtained by removing all the odd terms of the power series. Since $(-x)^{2n} + x^{2n} = 2x^{2n}$ but $(-x)^{2n+1} + x^{2n+1} = 0$, we have (just as before)

$$\frac{f(-x)+f(x)}{2} = \frac{1}{2}((a_0 - a_1 x + \frac{a_2}{2!}x^2 - \frac{a_3}{3!}x^3 + \cdots) + (a_0 + a_1 x + \frac{a_2}{2!}x^2 + \frac{a_3}{3!}x^3 + \cdots))$$
$$= a_0 + \frac{a_2}{2!}x^2 + \frac{a_4}{4!}x^4 + \cdots$$

as desired.

1.7.14. Observe that each "circular" subset S of $[n]$ counted by $g(n)$ falls into exactly one of the following cases.

1. S contains n, in which case $S - \{n\}$ is a subset of $\{2, 3, \ldots, n-2\}$ *arranged on a line* counted by $f(n-3)$ from Exercise 11.

2. S does not contain n, in which case S is a subset of $\{1, 2, \ldots, n-1\}$ *arranged on a line* counted by $f(n-1)$.

Hence, $g(n) = f(n-3) + f(n-1)$. We found an explicit formula for these numbers in Exercise 11. Hence, $g(n) = F_{n-1} + F_{n+1}$ where F_n is the nth Fibonacci number.

2.7.1. This is explained very nicely in the back of the book.

2.7.2. (a) Recall the power series $\sin x = \sum_{n\geq 0}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$ (or look it up in Equation (2.36)). If $f(x) = f_0 + f_1 x + f_2 x^2 + \cdots$ is the inverse of $\sin x$ then

$$x = f_0 + f_1 \sin x + f_2 (\sin x)^2 + \cdots$$

so comparing coefficients, we have

$$f_0 = 0, f_1 = 1, f_2 = 0, (\frac{-f_1}{3!} + f_3) = 0, \ldots$$

so

$$f(x) = x + \frac{1}{3!}x^3 + \cdots$$

(e) If $f(x) = f_0 + f_1 x + f_2 x^2 + \cdots$ is the inverse of $\log(1-x)$ then

$$x = \log(1 - f(x))$$

so

$$e^x = 1 - f(x)$$

so

$$f(x) = 1 - e^x = -x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \cdots$$