Homework 4

2.7.4. In each of these problems, let $f(x)$ denote the ogf of the given sequence.

(a) We have $\frac{1}{1-x} \overset{ogf}{\leftrightarrow} \{1\}$. Then by the polynomial rule, we have

$$(xD+7)\frac{1}{1-x} \overset{ogf}{\leftrightarrow} \{n+7\}$$

so

$$f(x) = (xD+7)\frac{1}{1-x} = \frac{x}{(1-x)^2} + \frac{7}{1-x}.$$

(b) Once we write out the formal power series, we can recognize it as a shifted geometric series.

$$x^4 + x^5 + x^6 + \cdots = x^4\frac{1}{1-x}.$$

(c) Once we write out the formal power series, we can recognize the closed form as a geometric series.

$$1 + x^2 + x^4 + x^6 + \cdots = 1 + (x^2) + (x^2)^2 + (x^2)^4 + \cdots = \frac{1}{1-x^2}.$$

(e) We have $e^x \overset{ogf}{\leftrightarrow} \{\frac{1}{n!}\}$. Then by the shift rule for ordinary generating functions, we have

$$\frac{1}{x^5}(e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!}) \overset{ogf}{\leftrightarrow} \{\frac{1}{(n+5)!}\}.$$

(f) We have

$$\frac{x}{1-x-x^2} \overset{ogf}{\leftrightarrow} \{F_n\}.$$

Therefore by the polynomial rule we obtain

$$(xD)\frac{x}{1-x-x^2} \overset{ogf}{\leftrightarrow} \{nF_n\}$$

which is $\frac{x(x^2+1)}{(1-x-x^2)^2}$.

2.7.6. We have

$$\frac{x}{(1-x)^2} \overset{ogf}{\leftrightarrow} \{n\}$$

so by the power rule, we obtain

$$(\frac{x}{(1-x)^2})^k \overset{ogf}{\leftrightarrow} \{\sum_{n_1+n_2+\cdots+n_k=n} n_1 n_2 \cdots n_k\}_{n\geq 0}.$$

Therefore, $f(n,k) = [x^n](\frac{x}{(1-x)^2})^k$ which by (2.39) is just

$$[x^n]x^k \sum_{n\geq 0} \binom{n+2k-1}{n} x^n$$

$$= [x^{n-k}] \sum_{n\geq 0} \binom{n+2k-1}{n} x^n$$

$$= \binom{(n-k)+2k-1}{(n-k)} = \binom{n+k-1}{n-k}.$$

2.7.7. We have

$$\frac{x^h}{1-x} \overset{ogf}{\leftrightarrow} \{1\}_{n\geq h}$$

so by the power rule, we obtain

$$(\frac{x^h}{1-x})^k \overset{ogf}{\leftrightarrow} \{\sum_{n_1+n_2+\cdots+n_k=n} a_{n_1} a_{n_2} \cdots a_{n_k}\}_{n\geq 0}$$

where the a_{n_i} are 0 if $n_i < h$ and 1 if $n_i \geq h$. Hence, we obtain

$$\sum_{n\geq 0} f(n,k,h)x^n = (\frac{x^h}{1-x})^k.$$

2.7.20. We have the formal power series identity

$$e^{t(x+y)} = e^{tx}e^{ty}.$$

Viewing these as exponential generating functions in t yields

$$\sum_{n\geq 0} (x+y)^n \frac{t^n}{n!} = (\sum_{n\geq 0} x^n \frac{t^n}{n!})(\sum_{t\geq 0} y^n \frac{t^n}{n!})$$

and applying the product rule for exponential generating functions to the right hand side, we obtain

$$\sum_{n\geq 0}(x+y)^n\frac{t^n}{n!} = \sum_{n\geq 0}(\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k})\frac{t^n}{n!}.$$

Taking the coefficient of $\frac{t_n}{n!}$ on both sides gives the desired identity. These coefficients must be equal because the formal power series are equal.

We could prove the multinomial theorem similarly by considering

$$e^{t(x_1+x_2+\cdots+x_k)} = e^{tx_1}e^{tx_2}\cdots e^{tx_k}$$

and using the product rule k times.