Homework 5

2.7.21. (a) First observe that even if $T = \{t_1, t_2, \ldots\}$ is just a *set* of integers, we can still create a formal power series out of it. We want a generating function that is 1 exactly on the elements in T and 0 everywhere else so let

$$T(x) = \sum_{t \in T} x^t = x^{t_1} + x^{t_2} + \cdots .$$

For example, if your set $T = \{1, 2, 3, 4\}$ then $T(x) = x + x^2 + x^3 + x^4$.

Next, use the power rule to transform $\{1\}_{n \in T} \overset{ogf}{\leftrightarrow} T(x)$ into

$$\sum_{n_1+n_2+\cdots+n_k=n \text{ with all } n_i \in T} 1 \cdot 1 \cdots 1 \overset{ogf}{\leftrightarrow} (T(x))^k.$$

In the example $T = \{1, 2, 3, 4\}$ with $k = 2$, we get $T(x)^k = x^2 + 2x^3 + 3x^4 + 4x^5 + 3x^6 + 2x^7 + x^8$ where $3x^4$ reflects the 3 ways of writing 4 as the sums of two numbers from T, namely $4 = 1 + 3 = 2 + 2 = 3 + 1$.

We can think of this generating function as the product

$$(x^{t_1} + x^{t_2} + \cdots)(x^{t_1} + x^{t_2} + \cdots) \cdots (x^{t_1} + x^{t_2} + \cdots)$$

where there are k factors. Each factor represents a *choice* of element from T.

(b) This problem will have a similar form as the last problem. We still want to choose k numbers from T but now we want to choose them *distinctly*. Hence, the generating function for all k's at once with no ordering is

$$(1 + x^{t_1})(1 + x^{t_2})(1 + x^{t_3}) \cdots .$$

There is one factor for each entry of T and each factor represents the choice between picking t_i or not. There are two problems to fix. We only want sums that have k parts. Hence, we need to introduce a variable y whose exponent records the number of parts of the sum. This gives

$$(1 + yx^{t_1})(1 + yx^{t_2})(1 + yx^{t_3}) \cdots$$

and the coefficient of y^k in this formal power series gives a generating function in x for the number of ways to write n as a sum of k numbers from T. However, this sum will not be ordered, so we also need to introduce a $k!$ to count all possible ways of writing the sum with those distinct terms. So in the end, we have

$$\sum_n g(n,k,T)x^n = [y^k]k!\prod_{t\in T}(1+yx^t).$$

For example, if $k=2$ and $T=\{1,2,3,4\}$ then

$$\sum_n g(n,k,T)x^n = [y^2]2!(1+yx)(1+yx^2)(1+yx^3)(1+yx^4) = 2(x^3+x^4+2x^5+x^6+x^7).$$

So the $4x^5$ term for example corresponds to $5 = 1+4 = 2+3 = 3+2 = 4+1$.

2.7.27. (a) We proved in Example 2.18 (and lecture) that the egf of D_n is

$$D(x) = e^{-x}\frac{1}{1-x}.$$

(b) Applying the rules for exponential generating functions, we see that

$$D_{n+1} = (n+1)D_n + (-1)^{n+1}$$

corresponds to

$$\frac{\partial}{\partial x}D(x) = (x\frac{\partial}{\partial x}+1)D(x) + \frac{\partial}{\partial x}e^{-x}$$

on the level of generating functions, and indeed

$$(1-x)D'(x) = D(x) + \frac{\partial}{\partial x}e^{-x}$$

by the product rule, after dividing both sides by $(1-x)$.

(c1) From the last problem, we get

$$D'(x) = D(x)(\frac{1}{1-x} - \frac{1-x}{1-x}) = D(x)\frac{x}{1-x}.$$

so we can differentiate this again to get

$$D''(x) = D'(x)\frac{x}{1-x} + D(x)(\frac{1}{1-x} + \frac{x}{(1-x)^2})$$

which proves that

$$(1-x)D''(x) = xD'(x) + D(x) + D'(x).$$

Now, applying the rules for exponential generating functions, we see that

$$D_{n+2} = (n+1)(D_{n+1} + D_n)$$

corresponds to

$$D''(x) = (x\frac{\partial}{\partial x} + 1)(D'(x) + D(x))$$

on the level of generating functions, and indeed this expands to

$$(1-x)D''(x) = xD'(x) + D(x) + D'(x)$$

which is what we showed above.

(c2) Equating the right hand sides of (b) and (c1) gives

$$nD_{n-1} = D_n + (-1)^{n+1}.$$

As in the recurrence we developed when we were solving this problem, the left hand side is the number of permutations of n with exactly one fixed point (because there are n ways to choose the fixed point, and D_{n-1} ways to fill in the rest of the permutation). Hence, the difference between the number of permutations with one fixed point and the number with no fixed points is exactly $(-1)^{n+1}$.

(d) If $D_k(n)$ is the number of permutations of n with exactly k fixed points then we have

$$D_k(n) = \binom{n}{k} D(n-k)$$

so

$$\sum_{n,k\geq 0} D_k(n)\frac{x^n y^k}{n!} = \sum_{n,k\geq 0} \binom{n}{k} D(n-k)\frac{x^n y^k}{n!}$$

which is an instance of the exponential multiplication rule. Specifically, we write the right hand side as

$$\sum_n (\sum_k \binom{n}{k} y^k D_{n-k})\frac{x^n}{n!}$$

which is the product of the corresponding *exponential* generating functions

$$e^{xy}\frac{e^{-x}}{1-x} = \frac{e^{x(y-1)}}{1-x}.$$

2.7.32. This is explained clearly in the back of the book. In part (d) we use the product rule for ordinary generating functions with equation (2.39).