# Homework 6

3.19.5. Recall that an involution is a permutation with all cycles having lengths 1 or 2. For example, $p = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{bmatrix}$ is an involution because in cycle notation $p = (13)(45)(2)$.

(a) We have that the exponential generating function for $\{T_n\}$ is $e^{x+x^2/2}$ from Section 3.8. Hence, applying the $xD\log$ operation as described in 1.6.1, we obtain

$$xD\log(\sum_{n\geq 0} T_n \frac{x^n}{n!}) = xD\log(e^{x+x^2/2})$$

$$\frac{\sum_{n\geq 0} nT_n \frac{x^n}{n!}}{\sum_{n\geq 0} T_n \frac{x^n}{n!}} = x(1+x^2)$$

$$\sum_{n\geq 0} nT_n \frac{x^n}{n!} = (x+x^2)\sum_{n\geq 0} T_n \frac{x^n}{n!}$$

and comparing coefficients of $x^n$ on each side yields

$$\frac{n}{n!}T_n = \frac{1}{(n-1)!}T_{n-1} + \frac{1}{(n-2)!}T_{n-2}.$$

Multiplying both sides by $(n-1)!$ gives the recurrence

$$T_n = T_{n-1} + (n-1)T_{n-2}.$$

(c) Since involutions have cycle lengths of 1 or 2, we must have that every involution $p$ of $n$ letters falls into exactly one of the following cases:

- $n$ is a fixed point of $p$, so $n$ appears in a cycle by itself.
- $n$ is not a fixed point of $p$, so $n$ appears in a cycle with exactly one other letter, say $i$.

In the first case, there are $T_{n-1}$ ways to complete the cycle $(n)$ to an involution of $n$. In the second case, there are $(n-1)$ choices for $i$ and $T_{n-2}$ ways to complete the cycle $(ni)$ to an involution of $n$. Hence, we obtain the recurrence

$$T_n = T_{n-1} + (n-1)T_{n-2}.$$

3.19.6. The deck enumerator for cycles with lengths $\geq 4$ is

$$D(x) = \sum_{n\geq 4}(n-1)!\frac{x^n}{n!} = \sum_{n\geq 4}\frac{x^n}{n} = \log\frac{1}{1-x} - (x + \frac{x^2}{2} + \frac{x^3}{3}).$$

Hence, by the exponential formula the 2-variable hand enumerator for permutations of $n$ with $k$ cycles of lengths $\geq 4$ is

$$\exp(yD(x)) = \exp(y(\log\frac{1}{1-x} - (x + \frac{x^2}{2} + \frac{x^3}{3}))) = \frac{1}{(1-x)^y}e^{-x-\frac{x^2}{2}-\frac{x^3}{3}}.$$

To obtain the generating function with respect to $n$ (without regard for the number of cycles), we set $y = 1$ in this formula.

3.19.12. Fix $k > 0$. The deck enumerator for cycles with lengths $\leq k$ is

$$D(x) = \sum_{n=1}^{k}(n-1)!\frac{x^n}{n!} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \frac{x^k}{k}.$$

Hence, by the exponential formula the 2-variable hand enumerator for permutations of $n$ with $m$ cycles of lengths $\leq k$ is

$$\exp(yD(x)) = e^{y(x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\frac{x^k}{k})}.$$

To obtain the generating function with respect to $n$ (without regard for the number of cycles), we set $y = 1$ in this formula.

Now, to obtain the generating function for the number of permutations of $n$ whose longest cycle actually has length $k$, we subtract the generating function for the number of permutations of $n$ with cycles of length $\leq (k-1)$. This yields

$$e^{(x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\frac{x^k}{k})} - e^{(x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\frac{x^{k-1}}{k-1})} = e^{(x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\frac{x^{k-1}}{k-1})}(e^{\frac{x^k}{k}} - 1).$$