Homework 7

3.19.10. Let $h(n,k)$ denote the Stirling number of the second kind. We found that $h(n,k)$ is equal to the coefficient of $\frac{x^n}{n!}$ in $\frac{1}{k!}(e^x-1)^k$, and using the Binomial theorem we obtain

$$\frac{1}{k!}(e^x-1)^k = \frac{1}{k!}\sum_{r=0}^{k}\binom{k}{r}(e^x)^r(-1)^{k-r} = \frac{1}{k!}\sum_{r=0}^{k}\binom{k}{r}(-1)^{k-r}(e^{rx})$$

$$= \frac{1}{k!}\sum_{r=0}^{k}\binom{k}{r}(-1)^{k-r}\sum_{n\geq 0} r^n\frac{x^n}{n!} = \sum_{n\geq 0}(\frac{1}{k!}\sum_{r=0}^{k}\binom{k}{r}(-1)^{k-r}r^n)\frac{x^n}{n!}.$$

Therefore, $h(n,k)$ is:

$$\frac{1}{k!}\sum_{r=0}^{k}\binom{k}{r}(-1)^{k-r}r^n = \sum_{r=1}^{k}\frac{(-1)^{k-r}r^n}{r!(k-r)!} = \sum_{r=1}^{k}\frac{(-1)^{k-r}r^{n-1}}{(r-1)!(k-r)!}$$

3.19.13 (a) Recall the exponential generating function $T(x) = e^{x+x^2/2} = \sum_{n\geq 0} t_n\frac{x^n}{n!}$ for the number t_n of involutions of n, and the exponential generating function $G(x) = \frac{e^{-x/2-x^2/4}}{\sqrt{1-x}} = \sum_{n\geq 0} g_n\frac{x^n}{n!}$ for the number g_n of 2-regular graphs on n vertices. Computing $T(x)G(x)G(x)$ yields $\frac{1}{1-x}$. Using the rules from Chapter 2, we can extract combinatorial meaning from this identity.

Viewing the geometric series as an *exponential* generating function corresponds to the sequence $\{n!\}_{n\geq 0}$. We can apply the exponential multiplication rule to the left side of the identity to obtain

$$\sum_{n_1+n_2+n_3=n}\frac{n!}{n_1!n_2!n_3!}t_{n_1}g_{n_2}g_{n_3} = n!$$

and dividing both sides by $n!$, we have

$$\sum_{n_1+n_2+n_3=n}\frac{1}{n_1!n_2!n_3!}t_{n_1}g_{n_2}g_{n_3} = 1.$$

The first version of the identity implies that there must be some way to view permutations as a pair of 2-regular graphs together with an involution. The rest of the problem (which was not assigned) explores this.

3.19.18. This is an unlabeled counting problem, so we'll use Theorem 3.21. The deck D_n consists of unlabeled paths and cycles with n vertices. For each n there is a unique path and a unique cycle. Moreover, for each $n \geq 3$, the path is distinct from the cycle. Hence, we have $d_1 = d_2 = 1$ and $d_n = 2$ for all $n \geq 3$.

By (3.34), the single variable ordinary hand enumerator for the number of not necessarily connected unlabeled graphs satisfying the conditions is

$$F(x) = \frac{1}{1-x}\frac{1}{1-x^2}\prod_{n\geq 3}\frac{1}{(1-x^n)^2}.$$

The problem asks us to relate this to the ordinary generating function

$$P(x) = \sum_{n\geq 0} p(n)x^n = \prod_{n\geq 1}\frac{1}{1-x^n}$$

for the number $p(n)$ of partitions of n. Comparing the two formulas, we have

$$F(x) = P(x)^2\frac{1}{1-x}\frac{1}{1-x^2}.$$