Homework 8

1. In each part below, find the ordinary generating function of the sequence $\{a_n\}_{n\geq 0}$ (where $a_0 = 1$ for each problem).

In these problems, we apply the restricted partition formula (3.51). Recall that W is the set of allowed multiplicities and R is the set of allowed parts.

(a) a_n is the number of partitions of n with all parts ≤ 4.

Here, we want $W = \{0, 1, 2, \ldots\}$ and $R = \{1, 2, 3, 4\}$. Hence, we obtain

$$\sum_{n\geq 0} a_n x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}.$$

(b) a_n is the number of partitions of n with largest part $= 4$.

We interpret a_n as the number of partitions of n with largest part ≤ 4 minus the number of partitions of n with largest part ≤ 3. Applying (a), we have

$$\sum_{n\geq 0} a_n x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)} - \frac{1}{(1-x)(1-x^2)(1-x^3)}.$$

Using a common denominator, this is

$$\frac{x^4}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$

which is just a shifted copy of (a). This is explained by the fact that there is a bijection between the partitions in (b) and the partitions in (a) given by removing the first part (which must have size 4).

(c) a_n is the number of partitions of n with no part appearing more than 2 times.

Here, we want $W = \{0, 1, 2\}$ and $R = \{1, 2, 3, \ldots\}$. Hence, we obtain

$$\sum_{n\geq 0} a_n x^n = \prod_{n\geq 1}(1 + x^n + x^{2n}).$$

(d) a_n is the number of partitions of n with no part divisible by 3.

Here, we want $W = \{0, 1, 2, \ldots\}$ and $R = \{1, 2, 4, 5, 7, 8, \ldots\}$. Hence, we obtain

$$\sum_{n\geq 0} a_n x^n = \prod_{n\geq 1, n \text{ not divisible by } 3} \frac{1}{1-x^n}$$
$$= \prod_{n\geq 1} \frac{1-x^{3n}}{1-x^n} = \prod_{n\geq 1} (1 + x^n + x^{2n})$$

which is the same as the generating function for (c). This is an example of Theorem 3.29.

(e) a_n is the number of partitions of n in which each odd part appears at most twice and each even part appears an even number of times.

We break the problem into two factors, each of which can be handled by Formula (3.51). The factor containing the odd parts is $\prod_{n \text{ odd}}(1 + x^n + x^{2n})$ and the factor containing the even parts is $\prod_{n \text{ even}}(1 + x^{2n} + x^{4n} + \cdots) = \prod_{n \text{ even}} \frac{1}{1-x^{2n}}$. Hence, we obtain

$$\sum_{n\geq 0} a_n x^n = \prod_{n \text{ odd}} (1 + x^n + x^{2n}) \prod_{n \text{ even}} \frac{1}{1-x^{2n}}.$$

2. Do problem 4.13.5 from the book.

This is clearly explained in the back of the book.

3. What is the number of essentially different *bracelets* that can be made from 4 beads with k different colors? Here we say that two bracelets are *essentially the same* if one can be rotated *or flipped over* to get the other.

(a) Begin by writing down the automorphism group G of all rotations and flips acting on the vertices of the square. (Hint: All of the rotations are still automorphisms, and we add 4 new flips to get 8 automorphisms in all.)

The list of rotations is

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} = (1)(2)(3)(4) \leftrightarrow x_1^4$$

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{bmatrix} = (0123) \leftrightarrow x_4$$

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \end{bmatrix} = (0321) \leftrightarrow x_4$$

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \end{bmatrix} = (02)(13) \leftrightarrow x_2^2$$

and the list of reflections is

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \end{bmatrix} = (01)(23) \leftrightarrow x_2^2$$

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 3 \end{bmatrix} = (02)(1)(3) \leftrightarrow x_1^2 x_2$$

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 2 & 1 \end{bmatrix} = (13)(0)(2) \leftrightarrow x_1^2 x_2$$

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix} = (03)(12) \leftrightarrow x_2^2$$

(b) Next, compute the cycle index Z_G of G.

Summing up the monomials we computed above, we obtain

$$Z_G(x_1, x_2, x_3, x_4) = \frac{1}{|G|} \sum_{p \in G} x_1^{z_1(p)} x_2^{z_2(p)} x_3^{z_3(p)} x_4^{z_4(p)}$$

$$= \frac{1}{8}(x_1^4 + 3x_2^2 + 2x_1^2 x_2 + 2x_4).$$

(c) Apply Pólya's theorem to get a formula for the number of square bracelets that use at most k colors.

The number of such bracelets is equal to the number of orbits of colorings of the square under the action of the rotations and reflections in G. By Pólya's theorem, we have that the number of these orbits is

$$Z_G(k, k, k, k) = \frac{1}{8}(k^4 + 3k^2 + 2k^3 + 2k).$$

For example, there are 21 such bracelets using at most 3 colors (as compared with 24 necklaces using at most 3 colors).