Final Exam

This is the final exam for Math 167, Fall 2007. The exam has 100 points, and you have 120 minutes to complete this exam. You may not use any notes or books, nor any calculating or computing devices. Please give *as much justification as you can* for all of your solutions.

Problem 1. (15 points)

Suppose there is a company in which every month half of those who are employed become rich, and a quarter of those who are rich become happy.

(a) Give the Markov transition matrix for this process.

(b) Find the steady state for this Markov process.

If we order the states as *employed, rich, happy* then the transition matrix is

$$\begin{bmatrix} 1/2 & 0 & 0 \\ 1/2 & 3/4 & 0 \\ 0 & 1/4 & 1 \end{bmatrix}$$

which is triangular. Consequently, the eigenvalues are $-\frac{1}{2}, \frac{3}{4}, 1$.

The steady state is given by the dominant eigenvector corresponding to eigenvalue 1. To obtain this, we consider the nullspace of

$$T - I = \begin{bmatrix} -1/2 & 0 & 0 \\ 1/2 & -1/4 & 0 \\ 0 & 1/4 & 0 \end{bmatrix}.$$

Since the eigenvalues are distinct we have that the corresponding eigenvectors are linearly independent. Therefore there is at most one eigenvector associated to eigenvalue 1. It is straightforward to verify that $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ lies in the nullspace of $T - I$ so it is an eigenvector with eigenvalue 1.

In the steady-state, everyone is happy.

Problem 2. (15 points)

(a) Give an example of a (2×2) matrix that has the vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ in its nullspace.

We need a matrix A so that the first column is -3 times the second column. So, we could take $A = \begin{bmatrix} -3a & a \\ -3b & b \end{bmatrix}$ for any values of a and b.

(b) Give an example of a (2×2) matrix that has the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as an eigenvector with eigenvalue 1.

We need a matrix A so that $(A - I)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. If we assume the form of the previous part, this means $\begin{bmatrix} -3a-1 & a \\ -3b & b-1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so we should choose a and b subject to the restriction $\begin{bmatrix} a \\ b-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hence, we should choose $a = 0$, and $b = 1$, so $A = \begin{bmatrix} 0 & 0 \\ -3 & 1 \end{bmatrix}$.

(c) Give an example of a (2×2) matrix that has $p(\lambda) = \lambda(\lambda - 1)$ as its characteristic polynomial.

Any triangular matrix with diagonal entries equal to 0 and 1 will do. Note that A has such a form.

(d) Give an example of a (2×2) matrix A that has all of the above properties at once and then compute $A^{101}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

We've been careful to build up the matrix $A = \begin{bmatrix} 0 & 0 \\ -3 & 1 \end{bmatrix}$ to have all of the properties. This means that $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector of A with eigenvalue 0, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 1.

Since we can write $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ in terms of the eigenvectors of A as $\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we can use linearity to calculate

$$A^{101}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = A^{101}(\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = A^{101}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2A^{101}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$(0)^{101}\begin{bmatrix}1\\3\end{bmatrix}+2(1)^{101}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\2\end{bmatrix}$$

Problem 3. (25 points) Let $A=\begin{bmatrix}1&2&-1\\0&1&2\end{bmatrix}$.

(a) Find the singular value decomposition $A=USV^t$.

We first form $AA^t=\begin{bmatrix}6&0\\0&5\end{bmatrix}$ and $A^tA=\begin{bmatrix}1&2&-1\\2&5&0\\-1&0&5\end{bmatrix}$. Since AA^t is already diagonal, we obtain $U=\begin{bmatrix}1&0\\0&1\end{bmatrix}$. We are also able to read off the singular values $\sqrt{6}$ and $\sqrt{5}$ from AA^t.

Since AA^t and A^tA have the same non-zero eigenvalues, we know what the eigenvalues of A^tA must be. We diagonalize A^tA by considering the nullspaces of $A^tA-\lambda_iI$ for $\lambda_1=6$, $\lambda_2=5$ and $\lambda_3=0$.

In the case of $\lambda_1=6$, we apply row reduction to find the nullspace of

$$\begin{bmatrix}-5&2&-1\\2&-1&0\\-1&0&-1\end{bmatrix}\to\begin{bmatrix}1&0&1\\0&-1&-2\\0&2&4\end{bmatrix}\to\begin{bmatrix}1&0&1\\0&1&2\\0&0&0\end{bmatrix}$$

and we find that $\begin{bmatrix}-1\\-2\\1\end{bmatrix}$ is an eigenvector for $\lambda_1=6$.

In the case of $\lambda_2=5$, we apply row reduction to find the nullspace of

$$\begin{bmatrix}-4&2&-1\\2&0&0\\-1&0&0\end{bmatrix}\to\begin{bmatrix}1&-1/2&1/4\\0&1&-1/2\\0&-1/2&1/4\end{bmatrix}\to\begin{bmatrix}1&-1/2&1/4\\0&1&-1/2\\0&0&0\end{bmatrix}$$

and we find that $\begin{bmatrix}0\\1\\2\end{bmatrix}$ is an eigenvector for $\lambda_2=5$.

In the case of $\lambda_3=0$, we apply row reduction to find the nullspace of

$$\begin{bmatrix}-1&2&-1\\2&5&0\\-1&0&5\end{bmatrix}\to\begin{bmatrix}1&2&-1\\0&1&2\\0&2&4\end{bmatrix}\to\begin{bmatrix}1&2&-1\\0&1&2\\0&0&0\end{bmatrix}$$

and we find that $\begin{bmatrix}5\\-2\\1\end{bmatrix}$ is an eigenvector for $\lambda_3=0$.

Since the eigenvectors correspond to distinct eigenvalues, they are orthogonal. We normalize them to obtain the orthogonal matrix V^t whose rows are the eigenvectors of A^tA. Thus,

$$V^t = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{5}{\sqrt{30}} & \frac{-2}{\sqrt{30}} & \frac{1}{\sqrt{30}} \end{bmatrix}$$

and it is easy to verify that $A = USV^t$ with

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & \sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{5}{\sqrt{30}} & \frac{-2}{\sqrt{30}} & \frac{1}{\sqrt{30}} \end{bmatrix}.$$

(b) Use your answer from (a) to find the pseudoinverse $A^+ = VS^+U^t$.

The pseudoinverse is

$$A^+ = \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{1}{\sqrt{5}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/6 & 0 \\ 2/6 & 1/5 \\ -1/6 & 2/5 \end{bmatrix}.$$

(c) Use your answer from (b) to find the minimal-length solution $x^+ = A^+b$ to

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \end{bmatrix}$$

In this problem, we have that b is in the column space of A and we use A^+ to project b into the row space of A. This gives the minimal length solution to $Ax = b$ because every solution can be decomposed as $x = x_n + x_r$ where x_n is in the nullspace of A and x_r is in the rowspace of A. Since the nullspace and the rowspace of any fixed matrix are orthogonal, we have that x_n and x_r are orthogonal so $||x||^2 = ||x_n||^2 + ||x_r||^2$. This shows that solutions in the row space minimize length.

Applying the formula, we have

$$A^+ \begin{bmatrix} 12 \\ 10 \end{bmatrix} = \begin{bmatrix} 1/6 & 0 \\ 2/6 & 1/5 \\ -1/6 & 2/5 \end{bmatrix} \begin{bmatrix} 12 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}.$$

(d) Show that your solution from (c) lies in the rowspace of A.

A basis for the rowspace of A is given by the rows of A and we can express x^+ as a linear combination of these vectors,

$$\begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Problem 4. (10 points) Suppose A is an $n \times n$ matrix which satisfies $A^4 = 0$.

(a) Find all of the eigenvalues of A.

(b) When is A diagonalizable? (Be as specific as possible.)

(Hint: You should give arguments which work for *any* matrix A that has the property $A^4 = 0$.)

We start by supposing that λ is an eigenvalue for A, so $Ax = \lambda x$ with $x \neq 0$. Then since $A^4 = 0$, we can apply both sides of this equation to the eigenvector x, so that $\lambda^4 x = 0$ which implies that λ must be 0 (since $x \neq 0$). So A has only the single eigenvalue 0.

If A were diagonalizable, then we would have $A = SDS^{-1}$, where D is the diagonal matrix of eigenvalues. But in our case, D must be the zero matrix because the only eigenvalue of A is $\lambda = 0$, which in turn means that A is the zero matrix. Hence, A is diagonalizable only when A actually equals zero. Note that there are lots of examples of non-zero nilpotent matricies however, for example we have

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and this matrix is not diagonalizable.

Problem 5. (10 points) Let U be an $n \times n$ complex matrix.

(a) Give the definition of a unitary matrix.

(b) Show that if U is unitary then $||Ux|| = ||x||$ for all x.

(c) Show that if λ is an eigenvalue of a unitary matrix U then $|\lambda| = 1$.

We have that U is unitary if it satisfies

$$U^*U = I = UU^*$$

where U^* denotes conjugate transpose. If U is unitary then for any vector x we can compute

$$||Ux||^2 = (Ux)^*(Ux) = x^*U^*Ux = x^*x = ||x||^2$$

and taking positive square roots gives (b). Therefore, if λ is an eigenvalue of a unitary matrix U, then from

$$Ux = \lambda x$$

we have

$$||Ux|| = |\lambda|||x||$$

and since the eigenvector $x \neq 0$, we obtain

$$|\lambda| = \frac{||Ux||}{||x||} = 1$$

from (b) as desired.

Problem 6. (15 points) Consider the following linear program.

Minimize $x_1 + 2x_2 - x_3$ subject to

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + x_4 \qquad &= 6 \\ 3x_1 + x_2 + 2x_3 \qquad + x_5 &= 6 \end{aligned}$$

(a) Verify that $P = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \\ 6 \end{bmatrix}$ is a basic feasible solution and determine its cost.

This is a basic solution because $m = 2$ of its variables are non-zero, and it is feasible because it satifies the equations with nonnegative components. The cost of P is 0.

(b) Determine which variables are the basic variables.

The non-zero variables are the basic variables and these are x_4, x_5.

(c) Which of the non-basic variables should enter?

The non-basic variable which best reduces cost should enter. This is x_3 because it has the most negative coefficient in the cost function.

(d) Which of the basic variables should leave?

The ratios

$$\frac{6}{3} < \frac{6}{2}$$

indicate that x_4 should leave, but even if we forgot this we could form the new solutions in each case. Since $\begin{bmatrix} 0 \\ 0 \\ 3 \\ -3 \\ 0 \end{bmatrix}$ has negative components, while $\begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}$ is feasible, we see that x_4 should leave.

(e) Find the new basic feasible solution and compute the cost at this new corner.

The new basic feasible solution is $\begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}$ with cost $-2 < 0$.

Problem 7. (10 points) Please circle TRUE or FALSE:

(a) (TRUE or FALSE) Every invertible matrix can be diagonalized.
This is false. Consider $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

(b) (TRUE or FALSE) Every diagonalizable matrix can be inverted.
This is false. Consider any matrix with eigenvalue 0, for example the zero matrix.

(c) (TRUE or FALSE) Every Hermitian matrix can be diagonalized by a unitary matrix.
This is true and known as the Spectral theorem.

(d) (TRUE or FALSE) If A is Hermitian and unitary then $A^2 = I$.
This is true because $A^* = A$ and $A^*A = I$, so $A^2 = I$.

(e) (TRUE or FALSE) If the eigenvalues of A are $1, 2, 3$ then $Ax = b$ has a unique solution.

This is true because A is diagonalizable with nonzero eigenvalues, hence invertible.