# Midterm Exam

This is the midterm exam for Math 167, Fall 2007. The exam has 100 points, and you have 50 minutes to complete this exam. You may not use any notes or books, nor any calculating or computing devices. Please give *as much justification as you can* for all of your solutions.

1. (30 points) Consider the $Ax = b$ system given by

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

(a) Give *two* descriptions of the column space $C(A)$ of $A$. One description should be a *basis* for $C(A)$. The other should be a set of *defining equations* that the components $y_1, y_2, y_3$ of any vector $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in C(A)$ must satisfy.

By definition, the column space $C(A)$ of $A$ is $\{b = Ax : x \in \mathbb{R}^3\}$ and row reducing the extended matrix gives

$$\begin{bmatrix} 1 & 1 & 1 & b_1 \\ 0 & 1 & 2 & b_2 + b_1 \\ 0 & 0 & 0 & b_3 + (b_2 + b_1) \end{bmatrix}$$

so the last row tells us that we must have $b_1 + b_2 + b_3 = 0$ in order for this system to be consistent. Hence, $C(A) = \{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} : b_1 + b_2 + b_3 = 0\}$.

Also, we have that $C(A)$ is the span of the columns of $A$. To reduce this to a basis, we remove $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ because this can be written in terms of the other two columns:

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

The remaining two vectors are linearly independent because if $x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ then we must have $x_1 = 0 = x_2$ (by considering the second and third components of the vectors). Hence, $C(A)$ has a basis that consists of the first two columns of $A$.

(b) When does the system $Ax = b$ have at least one solution?

This system has a solution when $b$ is in the column space of $A$. This means that $b_1 + b_2 + b_3$ must be 0, by part (a).

2. (30 points) Let $M_{2\times 2}$ be the vector space of all $2\times 2$ matrices. Let $T : M_{2\times 2} \to M_{2\times 2}$ be the matrix transpose function, so $T$ is defined by $T(A) = A^t$.

(a) Show that $T$ is a linear transformation.
We check that for any pair of $2\times 2$ matrices and $\lambda \in \mathbb{R}$ we have

$$T(\lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}) = (\lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix})^t$$

$$= \begin{bmatrix} \lambda a + e & \lambda b + f \\ \lambda c + g & \lambda d + h \end{bmatrix}^t = \begin{bmatrix} \lambda a + e & \lambda c + g \\ \lambda b + f & \lambda d + h \end{bmatrix}$$

$$= \lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix}^t + \begin{bmatrix} e & f \\ g & h \end{bmatrix}^t = \lambda T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) + T(\begin{bmatrix} e & f \\ g & h \end{bmatrix}).$$

(b) Give a basis for $M_{2\times 2}$.
Every vector in $M_{2\times 2}$ can be written uniquely as a linear combination of

$$e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

In fact, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ae_{11} + be_{12} + ce_{21} + de_{22}$. Hence, $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ forms a basis for $M_{2\times 2}$.

(c) Find a matrix for $T$ with respect to the basis you gave in (b).
Since we found 4 basis vectors for $M_{2\times 2}$ in (b), we are looking for a $4 \times 4$ matrix. The columns of the matrix are the result of applying $T$ to the basis vectors. We have $T(e_{11}) = e_{11}$ and $T(e_{22}) = e_{22}$ while $T(e_{12}) = e_{21}$ and $T(e_{21}) = e_{12}$. Hence, the matrix of $T$ with respect to the basis we chose in (b) is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(d) (BONUS: 10 points) Let $S = \{aI : a \in \mathbb{R}\}$ be the 1-dimensional subspace of $M_{2\times 2}$ that consists of multiples of the identity matrix. Find the matrix (with respect the the basis you gave in (b)) that projects "vectors" $A \in M_{2\times 2}$ onto the subspace $S$.

The projection to the line spanned by $I$ is given by the formula $\frac{II^t}{I^tI}$. With respect to the basis in (b), we write $I$ as the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Then, the matrix $\frac{II^t}{I^tI}$ is

$$\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Hence, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ projects to $\begin{bmatrix} \frac{1}{2}(a+d) & 0 \\ 0 & \frac{1}{2}(a+d) \end{bmatrix}$. In other words, the projection takes an arbitrary $2 \times 2$ matrix and sends it to the multiple of the identity that is the average of the diagonal values.

3. (30 points) Find the best least-squares fit for the model $y_i = at_i^2 + bt_i + c$ of the following data points:

| | | | | |
|---|---|---|---|---|
| $t$ | $-2$ | $-1$ | 1 | 2 |
| $y$ | 5 | 1 | 1 | 3 |

We first translate the model into a matrix equation $A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = y$ where $A = \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ and $y = \begin{bmatrix} 5 \\ 1 \\ 1 \\ 3 \end{bmatrix}$. By the theory of least-squares, the best solution for $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is given by solving the associated normal equations $A^t A \begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{bmatrix} = A^t y$. Row reducing, we obtain:

$$\begin{bmatrix} 34 & 0 & 10 & 34 \\ 0 & 10 & 0 & -4 \\ 10 & 0 & 4 & 10 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 10/34 & 1 \\ 0 & 1 & 0 & -2/5 \\ 1 & 0 & 2/5 & 1 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2/5 \\ 0 & 0 & 2/5 & 0 \end{bmatrix}$$

so the best fit model for the data is $y = t^2 - \frac{2}{5}t$.

4. (10 points) Please circle TRUE or FALSE:

(a) (TRUE or FALSE) If the columns of $A$ are linearly independent then $Ax = b$ has exactly one solution for every $b$.

This is false, because $b$ may not lie in the column space of $A$ when $A$ is not square. For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has no solution.

(b) (TRUE or FALSE) If $P$ is the matrix $A(A^tA)^{-1}A^t$ then $P = I$.

This is the formula for a projection matrix. When $A$ is not square, the projection matrix is not $I$ but still has the given form.

(c) (TRUE or FALSE) If $Ax = b$ is solvable for any $b$ and $A$ is an $n \times n$ matrix then the rowspace of $A$ is $\mathbb{R}^n$.

This is true, because the column space must be all of $\mathbb{R}^n$, so the nullspace is $\{0\}$. The rowspace is the orthogonal complement of the nullspace, so it must be all of $\mathbb{R}^n$.

(d) (TRUE or FALSE) If the column space of $A$ contains only the zero vector then $A$ is the zero matrix.

This is true, since if $A$ were not zero, then some column vector $b$ of $A$ would be nonzero, and $Ax = b$ would have a nontrival solution, implying that the column space of $A$ is nontrivial.

(e) (TRUE or FALSE) If $P$ is the matrix of a projection from $\mathbb{R}^n$ to a subspace $S$ then $P^3 = P$.

This is true, because projection matrices satisfy $P^2 = P$ (once we project to the subspace $S$ then any further projections have no effect because we're already on the subspace), so $P^n = P$ for all $n$.