HOMEWORK 1

1.4.10. (a) This is true, and follows from 1D(ii).
(b) This is false. For example,

$$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}0&0&0\\1&1&1\\0&0&0\end{bmatrix}=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

(c) This is true by 1D(iii).
(d) This is false. The correct statement is $(AB)^2 = ABAB$. A counterexample is

$$(\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix})^2=\begin{bmatrix}1&0\\0&0\end{bmatrix}\neq\begin{bmatrix}0&0\\0&0\end{bmatrix}=\begin{bmatrix}0&1\\0&0\end{bmatrix}^2\begin{bmatrix}0&0\\1&0\end{bmatrix}^2$$

1.4.32. (a) We represent this problem as the two linear equations $X = 2Y$ and $X + Y = 39$. In matrix notation, we have

$$\begin{bmatrix}1&-2\\1&1\end{bmatrix}\begin{bmatrix}X\\Y\end{bmatrix}=\begin{bmatrix}0\\39\end{bmatrix}$$

and after performing one step of Gaussian elimination, this reduces to

$$\begin{bmatrix}1&-2\\0&3\end{bmatrix}\begin{bmatrix}X\\Y\end{bmatrix}=\begin{bmatrix}0\\39\end{bmatrix}$$

which we solve to obtain $X = 26$ and $Y = 13$.

(b) We represent this problem as the two linear equations $5 = 2m+c$ and $7 = 3m + c$. In matrix notation, we have

$$\begin{bmatrix}2&1\\3&1\end{bmatrix}\begin{bmatrix}m\\c\end{bmatrix}=\begin{bmatrix}5\\7\end{bmatrix}$$

and after performing one step of Gaussian elimination, this reduces to

$$\begin{bmatrix}2&1\\0&-1/2\end{bmatrix}\begin{bmatrix}m\\c\end{bmatrix}=\begin{bmatrix}5\\-1/2\end{bmatrix}$$

which we solve to obtain $c = 1$ and $m = 2$.

1.4.36. No, these operations are not necessarily interchangable. Consider

$$EF\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&1\\0&1\end{bmatrix}\neq\begin{bmatrix}0&2\\0&1\end{bmatrix}=FE\begin{bmatrix}0&1\\0&0\end{bmatrix}.$$

1.4.38. The associative law states $(AB)C = A(BC)$ and if $AB = I = BC$ then this reduces to

$$(I)C = A(I)$$

which is

$$C = A.$$

1.5.4. (You should do all three of these, but I'll do the last one here.) To produce the LU factors for A we perform Gaussian elimination and keep track of the multipliers used at each step in the row reduction. We write $E_{i,j}(l)$ for the elementary matrix which subtracts l times row j from row i.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix} \to_{E_{3,1}(1)E_{2,1}(1)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 3 & 7 \end{bmatrix} \to_{E_{3,2}(1)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

Then, the LU factorization of A is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix}.$$

1.5.18. (You should do all three of these, but I'll do the first one here.) To find the number of solutions to these systems, we use Gaussian elimination. To reduce the first system written as an extended matrix, we first interchange rows:

$$\begin{bmatrix} 0 & 1 & -1 & 2 \\ 1 & -1 & 0 & 2 \\ 1 & 0 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 2 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$

Then, we can perform elimination steps to obtain

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The last row of the last extended matrix corresponds to the equation $0u + 0v + 0w = 2$ which has no solution. Hence, this system is singular with no solution.

1.5.22. The elimination moves $E_{3,2}(2)E_{3,1}(1)E_{2,1}(1)$ transform the coefficient matrix of the system into the upper triangular matrix $U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, so $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$. Observe that

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}$$

so $c = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$ does solve $Lc = b$. To solve $Ux = c$ we back-substitute obtaining $x_3 = 2$, $x_2 = 2 - 2x_3 = -2$ and $x_1 = 5 - x_2 - x_3 = 5$.

1.5.26. If $c = 2$ then the Gaussian elimination of A looks like

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 3 & 5 & 1 \end{bmatrix}$$

which has a zero in the second pivot position. In this case, we must apply a permutation matrix to interchange the second and third rows, so $A = LU$ is not possible. If $4 - 2c = 5 - 3c$, or equivalently $c = 1$, then the Gaussian elimination of A looks like

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which has a zero in the third pivot position.