Homework 2

1.6.14. Suppose B is square and form $A = B + B^t$, $K = B - B^t$. Then,

$$(B + B^t)^t = B^t + (B^t)^t = B^t + B$$

which proves that A is symmetric and

$$(B - B^t)^t = B^t - (B^t)^t = -(B - B^t)$$

which proves that K is skew-symmetric. If $B = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$ then

$$A = B + B^t = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$$

and

$$K = B - B^t = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

Observe that we can always recover B from A and K by

$$B = \frac{1}{2}((B + B^t) + (B - B^t))$$

and in this case

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

1.6.42. To prove that A is invertible, we apply the Gauss-Jordan method to the matrix.

$$\begin{bmatrix} a & b & b & 1 & 0 & 0 \\ a & a & b & 0 & 1 & 0 \\ a & a & a & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} a & b & b & 1 & 0 & 0 \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & a-b & a-b & -1 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} a & b & b & 1 & 0 & 0 \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & 0 & a-b & 0 & -1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} a & 0 & 0 & 1 + \frac{b}{a-b} & \frac{-b}{a-b} + \frac{b}{a+b} & \frac{-b}{a-b} \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & 0 & a-b & 0 & -1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{a}(1 + \frac{b}{a-b}) & 0 & \frac{-b}{a(a-b)} \\ 0 & 1 & 0 & -\frac{1}{a-b} & \frac{1}{a-b} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{a-b} & \frac{1}{a-b} \end{bmatrix}$$

Hence, if $a \neq 0$ and $a - b \neq 0$, then A^{-1} exists:

$$A^{-1} = \begin{bmatrix} \frac{1}{a}(1 + \frac{b}{a-b}) & 0 & \frac{-b}{a(a-b)} \\ -\frac{1}{a-b} & \frac{1}{a-b} & 0 \\ 0 & -\frac{1}{a-b} & \frac{1}{a-b} \end{bmatrix}$$

1.6.56. Using the rules for transpose and the facts that $A^t = A$, $B^t = B$, we have:
(a) $(A^2 - B^2)^t = (A^2)^t - (B^2)^t = (A^t)^2 - (B^t)^2 = A^2 - B^2$ shows that $A^2 - B^2$ is symmetric.
(b) $((A+B)(A-B))^t = (A-B)^t(A+B)^t = (A^t - B^t)(A^t + B^t) = (A-B)(A+B)$ shows us that we can expect a counterexample if $(A-B)$ and $(A+B)$ do not commute. For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ so $A + B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $A - B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.
(c) $(ABA)^t = A^t B^t A^t = ABA$ shows that ABA is symmetric.
(d) $(ABAB)^t = B^t A^t B^t A^t = BABA$ shows us that we can expect a counterexample if A and B do not commute.

1.6.60. (You should do all three of these but I'll do the first one here.) We subtract 3 times row 1 from row 2 to obtain

$$\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & -7 \end{bmatrix}$$

so

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

which has the form LDL^t.

2.1.10. In the space $M_{2\times 2}$ of 2 by 2 matrices, the zero vector is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (because when we add this matrix to any 2 by 2 matrix B, we get B again). The vector $\frac{1}{2}A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ (because scalar multiplication on matrices is defined componentwise) and $-A = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$ (because this is the unique vector such that $A + (-A) = 0$). The smallest subspace of $M_{2\times 2}$ that contains A must contain the set S of all scalar multiples of A. If cA and dA are two vectors in S where $c, d \in \mathbb{R}$, we have

$$cA + dA = (c + d)A$$

which is in S since it is another scalar multiple of A. Hence, S is a subspace, so it is the smallest subspace containing A.

2.1.16. The plane P_0 is parallel to the plane P if and only if there are no points (x, y, z) on both of them. This occurs precisely when the linear system that consists of the defining equations for P and P_0 is inconsistent. This in turn occurs precisely when the defining equation for P_0 is of the form $x + y - 2z = c$ where $c \neq 4$. Since P_0 should go through the origin, we must have $c = 0$. If there are two points (x, y, z) and (x', y', z') satisfying

$$x + y - 2z = 0 = x' + y' - 2z'$$

then $(x + x') + (y + y') - 2(z + z') = 0$, so $(x + x', y + y', z + z')$ is on P_0. For example, take $(x, y, z) = (1, 1, 1)$ and $(x', y', z') = (1, 3, 2)$ so $(x + x', y + y', z + z') = (2, 4, 3)$.

2.1.22. (You should do both of these but I'll do the second one here.) Perform Gaussian elimination on the rows of the extended matrix, keeping track of the results on the b vector to obtain

$$\begin{bmatrix} 1 & 4 & b_1 \\ 2 & 9 & b_2 \\ -1 & -4 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & b_3 + b_1 \end{bmatrix}.$$

The third row expresses the equation $0 = 0x_1 + 0x_2 = b_3 + b_1$, so we must have $b_1 = -b_3$ in order for the system to be solvable.

2.1.30. Recall that $Ax = b$ is solvable if and only if b can be expressed as a linear combination of the columns of A. Hence, if the 9×12 system $Ax = b$ is solvable for every b then $C(A) = \{Ax : x \in \mathbb{R}^{12}\}$ must be all of $\mathbb{R}^9$.

2.2.30. We row reduce A to obtain

$$\begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 2 & 5 & 7 & 6 & b_2 \\ 2 & 3 & 5 & 2 & b_3 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & -1 & -1 & -2 & b_3 - b_1 \end{bmatrix}$$

$$\to \begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_1 + (b_2 - b_1) \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 1 & -2 & \frac{1}{2}b_1 - 2(b_2 - b_1) \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_1 + (b_2 - b_1) \end{bmatrix}$$

so by the last equation, b_3 must be $2b_1 - b_2$ or there is no solution. This gives a description of the column space $C(A) = \{x \in \mathbb{R}^3 : x_3 + x_2 - 2x_1 = 0\}$.

Since there are no pivots in columns 3 or 4, we have that x_3 and x_4 are free variables. To describe the nullspace $N(A)$, we solve $Ax = 0$. Setting exactly one of the free variables to 1, one at a time, we have $x_2 = -1$ and $x_1 = -1$ when $x_3 = 1, x_4 = 0$. Similarly, $x_2 = -2$ and $x_1 = 2$ when $x_3 = 0, x_4 = 1$. Hence, $N(A)$ is the span of

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

To solve $Ax = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$, consider the corresponding row reduced matrix

$$\begin{bmatrix} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so $x_1 = 4 - x_3 + 2x_4$ and $x_2 = -1 - x_3 - 2x_4$. When the free variables are zero, this gives $x_p = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, and $x_p + x_n$ is also a solution for any $x_n \in N(A)$.