Homework 3

2.3.10. We can think of this subspace as the vectors that are perpendicular to $v = \begin{bmatrix} 1 \\ 2 \\ -3 \\ -1 \end{bmatrix}$ since taking the dot product of any vector in $\mathbb{R}^4$ with v gives 0 by the defining equation. Hence, this subspace is the nullspace of the matrix

$$\begin{bmatrix} 1 & 2 & -3 & -1 \end{bmatrix}$$

and if we view this as being in row reduced form, then y, z and t are free variables. Hence, three independent vectors in the nullspace are:

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

and this is the maximum number of independent vectors since the dimension of the nullspace equals the number of free variables in the system.

2.3.26. (a) This is true by statement 2L.

(b) This is not true. Suppose $B = \{v_1, v_2, \ldots, v_6\}$ is a basis for $\mathbb{R}^6$. Then we can obtain *at most six* different subspaces by removing one vector from B. But there are *infinitely* many 5-dimensional subspaces of $\mathbb{R}^6$. If S is not one of the six subspaces that we got by removing a vector from B in all possible ways, then we have a counterexample to the statement.

2.6.14. If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is linear then $T(av + w) = aT(v) + T(w)$. Since $T(v)$ and $T(w)$ are just vectors in $\mathbb{R}^3$, we have

$$T(T(av + w)) = T(aT(v) + T(w)) = aT(T(v)) + T(T(w))$$

so T^2 is linear.

2.6.22. (a) This is linear because T is multiplication by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(b) This is linear because T is multiplication by the matrix $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

(c) This is linear because T is multiplication by the matrix $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

(d) This is not linear because $T(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

2.6.36. (a) The columns of the matrix are the result of transforming the basis vectors. Hence, the matrix is

$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}.$$

(b) The matrix of this transformation is the inverse of the matrix from (a). Hence, it is

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

using the Gauss-Jordan method (not shown).

(c) Matrix multiplication is linear so if a matrix T transforms $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then it *must* transform $\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ to $\frac{1}{2}T(\begin{bmatrix} 2 \\ 6 \end{bmatrix}) = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$ and this is not $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

3.1.8. If $x \in V \cap W$ then $x \in V$ and $x \in W$ and so $x^t x = 0$ since V and W are orthogonal. But the only vector with length 0 is $x = 0$.

3.1.20. The statement $(S^\perp)^\perp = S$ means "S is the orthogonal complement of $S^\perp$." It follows from the statement that if S and W are subspaces and $W = S^\perp$ then $S = W^\perp = (S^\perp)^\perp$.

3.1.38. If $S = \{0\}$ then $S^\perp = \{v \in \mathbb{R}^3 : v^t s = 0 \text{ for all } s \in S\} = \mathbb{R}^3$. If $S = span\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\}$ then $S^\perp = \{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x + y + z = 0\}$ which is a plane in $\mathbb{R}^3$. If $S = span\{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}\}$ then $S^\perp = \{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 2x = 0$ and $3y = 0\}$ which is the line spanned by $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

3.2.10. If P is a projection matrix given by $aa^t/a^t a$ for some $a \in \mathbb{R}^n$ then P is invertible only if $n = 1$ (and the projection matrix is just

1×1 otherwise known as multiplication by a scalar). In the general case, P will be sending $\mathbb{R}^n$ onto a 1-dimensional subspace which cannot be invertible. One way to see this is to note that the row space of P is 1-dimensional (because each row is a multiple of a^t) so the nullspace of P must be $n-1$ dimensional (because the row space and the nullspace are othogonal complements). But we know that P is invertible only if $N(P) = \{0\}$ so P is not invertible when $n > 1$.