HOMEWORK 4

3.4.14. From $a = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $b = \begin{bmatrix}1\\0\\1\end{bmatrix}$ and $c = \begin{bmatrix}0\\1\\1\end{bmatrix}$, we first find orthogonal vectors v_1, v_2, v_3 in stages using Gram-Schmidt. Take

$$v_1 = a = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

and

$$v_2 = b - \frac{b^t v_1}{v_1^t v_1} v_1 = \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

and

$$v_3 = c - \frac{c^t v_1}{v_1^t v_1} v_1 - \frac{c^t v_2}{v_2^t v_2} v_2 = \begin{bmatrix}0\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{1/2}{3/2}\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix} = \begin{bmatrix}-2/3\\2/3\\2/3\end{bmatrix}.$$

Normalizing these, we have

$$q_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}, q_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\end{bmatrix}, q_3 = \frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\1\end{bmatrix}.$$

3.4.16. To perform Gram-Schmidt on these two vectors, we just normalize a_1 to obtain

$$q_1 = \frac{1}{\sqrt{9}}\begin{bmatrix}1\\2\\2\end{bmatrix}$$

then calculuate

$$q_2' = a_2 - (a_1^t q_1) q_1 = \begin{bmatrix}1\\3\\1\end{bmatrix} - \frac{9}{9}\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}0\\1\\-1\end{bmatrix}$$

and normalizing gives

$$q_2 = -\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\end{bmatrix}.$$

As an $A = QR$ factorization, this is

$$\begin{bmatrix}1 & 1\\2 & 3\\2 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & 0\\ \frac{2}{3} & \frac{1}{\sqrt{2}}\\ \frac{2}{3} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}3 & 3\\0 & \sqrt{2}\end{bmatrix}.$$

In general, when there are n vectors a_i with m components, then A and Q are $m \times n$ matrices and R is $n \times n$.

3.4.32. (a) We can view S as the nullspace of $\begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix}$ so a basis for this space is obtained by setting exactly one of the free variables to 1, one at a time. In this case we obtain

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) Since $dim(S) + dim(S^{\perp}) = n = 4$, we need to find a single vector that is orthogonal to the three vectors from (a). But S was defined precisely by the condition that taking the dot product of any vector in S with $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ is 0. Hence, $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ is a basis for $S^{\perp}$.

(c) The most straightforward way to do this is to solve the associated $Ax = b$ problem to write b as a linear combinations of the columns of A, which in this case will be the basis vectors we found in (a) and (b). The row reduction steps are

$$\begin{bmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & -4 & -2 \end{bmatrix}$$

so back-substituting gives $x_4 = 1/2$, $x_3 = 3/2$, $x_2 = 1/2$ and $x_1 = 1/2$. Hence,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = ((1/2) \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + (1/2) \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + (3/2) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}) + (1/2) \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

is an expression of the desired form.

We could also project $b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ to $S^{\perp}$ using the line projection formula, and then project b to the column space of $A = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ using the generalized projection formula given in section 3.3.

3.3.8. Since P projects points of $\mathbb{R}^n$ to the subspace S, the column space of P is by definition $\{Px : x \in \mathbb{R}^n\} = S$. The rank of P is the dimension of the column space, which is k.

3.3.14. The projection matrix to the column space of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ is given by $P = A(A^tA)^{-1}A^t$ which is

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Any vector in the nullspace of this matrix will be projected to zero. For example, the nullspace contains the nonzero vector $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

3.3.24. We want to fit the given data to the equation of a line $y_i = mt_i + b$ where m and b are our unknown parameters to be estimated and (y_i, t_i) are our data points. Thus we want the least squares solution to $Ax = b$ given by

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix}.$$

The normal equations are $A^t A\hat{x} = A^t b$ given by

$$\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \hat{m} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} -15 \\ -6 \end{bmatrix}$$

and this system can be solved in the usual way. Row reduction gives

$$\begin{bmatrix} 6 & 2 & -15 \\ 0 & 10/3 & -1 \end{bmatrix}$$

so $\hat{b} = -3/10$ and $\hat{m} = -144/60$. This means that the given data is best approximated by the line $y = -\frac{144}{60}t - \frac{3}{10}$.