Midterm Exam 1

This is the midterm exam for Math 16B, Winter 2009. Please write your name clearly at the top of the exam. The exam has 100 points, and you have 50 minutes to complete this exam. You may not use any notes or books, nor any calculating or computing devices. Please write your answers as simply as you can, and show the steps you use to obtain your solution.

1. (18 points) Solve each equation for x.

(a) $4e^{3-x} - 7 = 2$.

Adding 7 to both sides and dividing by 4, we have

$$e^{3-x} = \frac{9}{4}.$$

Taking ln of both sides yields

$$3 - x = \ln \frac{9}{4}$$

and we can solve for x obtaining

$$x = 3 - \ln \frac{9}{4} = 3 - \ln 9 + \ln 4.$$

(b) $\frac{4}{1+3e^{1-1/(3x)}} = 1$.

Multiplying both sides by the denominator of the left hand side, we have

$$4 = 1 + 3e^{1-1/(3x)}.$$

Subtracting 1 from both sides, we have $3 = 3e^{1-1/(3x)}$ so

$$1 = e^{1-1/(3x)}.$$

Since $e^0 = 1$ and e^x is a 1-1 function, we must have

$$1 - \frac{1}{3x} = 0$$

so $\frac{1}{3x} = 1$ and $3x = 1$ whence

$$x = \frac{1}{3}.$$

(c) $5^x = e^5$.

This is a change of base formula. We can solve it by rewriting 5^x as $e^{x \ln 5}$ so

$$e^{x \ln 5} = e^5.$$

Since e^x is a 1-1 function, we must then have

$$x \ln 5 = 5$$

so

$$x = \frac{5}{\ln 5}.$$

2. (20 points) Let $A(t)$ be the value of an account that is compounded continuously, so $A(t)$ grows exponentially. Suppose that after 10 years, the value of the account has tripled. What is the interest rate of the account?

Since $A(t)$ grows exponentially, we know that $A(t)$ has the form Pe^{rt} for some parameters P and r. Our goal is to find r given

$$\begin{aligned} A(10) &= 3A(0) \\ Pe^{10r} &= 3Pe^{0r} \\ Pe^{10r} &= 3P \\ e^{10r} &= 3 \\ 10r &= \ln 3 \end{aligned}$$

so

$$r = \frac{1}{10}\ln 3$$

(It turns out that this is about 11%.)

3. (18 points) Find the derivative of each function.

(a) $f(x) = \ln\sqrt{\frac{(1+x)^2}{1-x^2}}$.

We begin by simplifying the function. We can rewrite $f(x)$ as

$$\ln\left(\frac{(1+x)(1+x)}{(1-x)(1+x)}\right)^{1/2}$$

$$= \ln\left(\frac{1+x}{1-x}\right)^{1/2} = \frac{1}{2}\ln\frac{1+x}{1-x} = \frac{1}{2}(\ln(1+x) - \ln(1-x)).$$

This version of $f(x)$ is much easier to differentiate. Using the linearity of the derivative, we have

$$\begin{aligned} f'(x) &= \frac{1}{2}(\frac{d}{dx}\left[\ln(1+x)\right] - \frac{d}{dx}\left[\ln(1-x)\right]) \\ &= \frac{1}{2}(\frac{1}{1+x}\frac{d}{dx}\left[1+x\right] - \frac{1}{1-x}\frac{d}{dx}\left[1-x\right]) \\ &= \frac{1}{2}(\frac{1}{1+x} + \frac{1}{1-x}) \end{aligned}$$

(b) $g(x) = (1+\sqrt{x})^{1-x}$.

We first rewrite $g(x)$ as $e^{(1-x)\ln(1+\sqrt{x})}$ and then use the rule for differentiating functions of the form $e^{u(x)}$ to obtain

$$\begin{aligned} g'(x) &= e^{(1-x)\ln(1+\sqrt{x})}\frac{d}{dx}\left[(1-x)\ln(1+\sqrt{x})\right] \\ &= (1+\sqrt{x})^{1-x}\left(-\ln(1+\sqrt{x}) + (1-x)\frac{d}{dx}\left[\ln(1+x^{1/2})\right]\right) \\ &= (1+\sqrt{x})^{1-x}\left(-\ln(1+\sqrt{x}) + (1-x)\frac{1}{1+x^{1/2}}\frac{d}{dx}\left[1+x^{1/2}\right]\right) \\ &= (1+\sqrt{x})^{1-x}\left(-\ln(1+\sqrt{x}) + (1-x)\frac{1}{1+x^{1/2}}\frac{1}{2}x^{-1/2}\right) \\ &= (1+\sqrt{x})^{1-x}\left(\frac{1-x}{2(\sqrt{x}+x)} - \ln(1+\sqrt{x})\right) \end{aligned}$$

4. (20 points) Find a function f that satisfies $f''(t) = -32$, $f'(0) = 8$ and $f(2) = 0$.

We begin by antidifferentiating $f''(t) = -32$ obtaining

$$f'(t) = -32t + C$$

for some constant C and since $f'(0) = 8$ we have $C = 8$. Then, antidifferentiating $f'(t) = -32t + 8$ we obtain

$$f(t) = -16t^2 + 8t + D$$

for some constant D and since $f(2) = 0$, we have $0 = -64 + 16 + D$ so $D = 48$. Hence,

$$f(t) = -16t^2 + 8t + 48.$$

5. (24 points) Find each integral. If you use any substitutions, please write your final answer in terms of the original variable x.

(a) $\int 2x^2(1+x^3)^2\,dx$

Here we aim to apply the power rule, and make the substitution $u = 1 + x^3$ so $du = 3x^2\,dx$ and then multiplying both sides by $2/3$, we obtain $2x^2\,dx = \frac{2}{3}\,du$. Next, we can rewrite the integral as

$$\int u^2\,(\frac{2}{3}\,du) = \frac{2}{3}\int u^2\,du = \frac{2}{3}\frac{u^3}{3} + C = \frac{2}{9}(1+x^3)^3 + C.$$

(b) $\int \frac{e^x + e^{-x}}{(e^x - e^{-x})^2}\,dx$

Here we look to apply the power rule, making the substitution $u = e^x - e^{-x}$ so $du = e^x + e^{-x}\,dx$. Then, we can rewrite the integral as

$$\int \frac{1}{u^2}\,du = \int u^{-2}\,du = \frac{u^{-1}}{-1} + C = -\frac{1}{e^x - e^{-x}} + C.$$

(c) $\int 6e^{-3x} + 5\,dx$

We begin by splitting the integral into the sum of two simpler integrals $\int 6e^{-3x}\,dx + \int 5\,dx$. The second term is just $5x + C$. For the first integral, we want to apply the exponential rule, so we make the substitution $u = -3x$ yielding $du = -3\,dx$ and so $6\,dx = -2\,du$ and we rewrite the integral as

$$-2\int e^u\,du + (5x + C) = -2(e^u + C) + (5x + C) = -2e^{-3x} + 5x + C.$$