Midterm Exam 2

This is the second midterm exam for Math 16B, Winter 2009. Please write your name clearly at the top of the exam. The exam has 100 points, and you have 50 minutes to complete this exam. You may not use any notes or books, nor any calculating or computing devices. Please give *as much justification as you can* for all of your solutions.

1. (14 points) Find the area of the region bounded by the graphs of $f(x) = -x^2 + x$ and $g(x) = x^3 - x^2$.

We begin by determining the points where the curves intersect by solving $f(x) = g(x)$ for x. We obtain

$$-x^2 + x = x^3 - x^2$$
$$0 = x^3 - x$$

so

$$0 = x(x+1)(x-1)$$

and the curves intersect at $x = -1$, $x = 0$, and $x = 1$. Hence, the region bounded by the curves lies on the intervals $[-1, 0]$ and $[0, 1]$.

Next, we determine which of the functions is larger on each interval. For x in $(-1, 0)$, we can take a representative value, say $x = -\frac{1}{2}$, to find that

$$f(-\frac{1}{2}) = -\frac{1}{4} - \frac{1}{2} < -\frac{1}{8} - \frac{1}{4} = g(-\frac{1}{2})$$

so $f(x) < g(x)$ on $(-1, 0)$. For x in $(0, 1)$, we can take a representative value, say $x = \frac{1}{2}$, to find that

$$f(1/2) = -(1/4) + (1/2) > 0 > (1/8) - (1/4) = g(1/2)$$

so $f(x) > g(x)$ on $(0, 1)$.

Hence, we have that the area bounded by the curves is given by

$$\int_{-1}^{0} g(x) - f(x)\, dx + \int_{0}^{1} f(x) - g(x)\, dx = \int_{-1}^{0} x^3 - x\, dx - \int_{0}^{1} x^3 - x\, dx$$

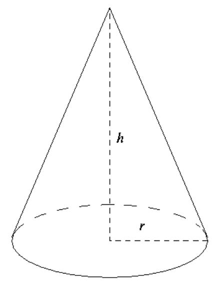
$$\left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} - \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{0}^{1}$$

$$\left[0 - \left(\frac{(-1)^4}{4} - \frac{(-1)^2}{2}\right)\right] - \left[\left(\frac{1^4}{4} - \frac{1^2}{2}\right) - 0\right]$$

$$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

which is positive, consistent with the interpretation as an area.

2. (14 points) Find the volume of a right cone with height h and radius r, as a solid of revolution.



We can obtain the cone as a solid of revolution by revolving the region bounded by the x-axis, the line $y = \frac{r}{h}x$, and the vertical line $x = h$, about the x-axis. (This region forms a right triangle.) By the disk method, the volume is given by

$$\pi \int_0^h (\frac{r}{h}x)^2 \, dx = \pi \frac{r^2}{h^2} \left[\frac{x^3}{3} \right]_0^h = \pi \frac{hr^2}{3}.$$

3. (12 points) Evaluate $\int \ln(x^3) \, dx$.

Applying integration by parts, let $dv = dx$ and $u = \ln x^3$ so $v = x$ and $du = x^{-3} 3x^2 \, dx = \frac{3}{x} \, dx$. Then, applying the integration by parts formula, the integral becomes

$$\begin{aligned} & x \ln x^3 - \int x \frac{3}{x} \, dx \\ &= x \ln x^3 - 3x + C. \\ &= 3(x \ln x - x) + C. \end{aligned}$$

4. (12 points) Evaluate $\int \frac{\sin(\ln x) e^{\cos(\ln x)}}{x} + 2 \sin(x) \, dx$.

We first split the integral as

$$\int \frac{\sin(\ln x) e^{\cos(\ln x)}}{x} \, dx + 2 \int \sin(x) \, dx.$$

The second summand is $-2\cos(x) + C$. For the first summand, we apply substitution with $u = \cos(\ln x)$ so $du = -\sin(\ln x) \frac{1}{x} \, dx$ and we rewrite the integral as

$$-\int e^u \, du - 2\cos(x) + C = -e^u - 2\cos(x) + C.$$

This expression has mixed variables, so we rewrite the expression in terms of x as

$$-e^{\cos(\ln x)} - 2\cos(x) + C.$$

5. (12 points) Evaluate $\int x^2 \sin x \, dx$.

We apply integration by parts with $dv = \sin x \, dx$ and $u = x^2$ so $v = -\cos x$ and $du = 2x \, dx$. The integral becomes

$$-x^2 \cos x + \int 2x \cos x \, dx = -x^2 \cos x + 2 \int x \cos x \, dx.$$

Then, we apply integration by parts again with $dv = \cos x \, dx$ and $u = x$ so $v = \sin x$ and $du = dx$. The integral becomes

$$\begin{aligned} &-x^2 \cos x + 2\left(x \sin x - \int \sin x \, dx\right) \\ &= -x^2 \cos x + 2x \sin x + 2\cos x + C \\ &= (2 - x^2)\cos x + 2x \sin x + C. \end{aligned}$$

6. (12 points) Evaluate $\int_0^1 \frac{1}{x^2+3x+2} \, dx$.

We apply partial fractions to write the integrand as $\frac{A}{x+2} + \frac{B}{x+1}$, so the basic equation is

$$1 = A(x+1) + B(x+2).$$

Pluging in $x = -1$, we obtain $B = 1$ and pluging in $x = -2$, we get $A = -1$. Hence, the integral becomes

$$\begin{aligned} &\int_0^1 \frac{1}{x+1} - \frac{1}{x+2} \, dx \\ &= \left[\ln|x+1| - \ln|x+2|\right]_0^1 \\ &= \ln 2 - \ln 3 - (\ln 1 - \ln 2) = 2\ln 2 - \ln 3 = \ln \frac{4}{3}. \end{aligned}$$

7. (12 points) Evaluate $\int x \sqrt[3]{2x-1} \, dx$.

Applying substitution, we let $u = 2x - 1$ so $du = 2\,dx$ and $x = \frac{1}{2}(u+1)$. Then, we can rewrite the integral in terms of u and du as

$$\begin{aligned} &\frac{1}{4}\int (u+1)u^{1/3} \, du = \frac{1}{4}\int u^{4/3} + u^{1/3} \, du = \frac{1}{4}\left(\frac{u^{7/3}}{7/3} + \frac{u^{4/3}}{4/3} + C\right) \\ &= \left(\frac{3}{28}(2x-1)^{7/3} + \frac{3}{16}(2x-1)^{4/3}\right) + C. \end{aligned}$$

8. (12 points) Evaluate $\int_0^1 e^{-2\sqrt{x}}\,dx$.

We substitute $w = -2\sqrt{x}$ so $dw = -x^{-1/2}\,dx$ so $dx = \frac{1}{2}w\,dw$. The limits of integration become $w = 0$ and $w = -2$. Hence, we can rewrite the integral as

$$\int_0^{-2} \frac{1}{2} we^w\,dw = \frac{1}{2}\int_0^{-2} we^w\,dw$$

We solve this using integration by parts with $dv = e^w\,dw$ and $u = w$ so $v = e^w$ and $du = dw$, obtaining

$$\frac{1}{2}\left[we^w - \int e^w dw\right]_0^{-2}$$

$$\frac{1}{2}\left[we^w - e^w\right]_0^{-2}$$

$$\frac{1}{2}\left(-2e^{-2} - e^{-2} + 1\right) = -\frac{3}{2}e^{-2} + \frac{1}{2}.$$