Homework 2

4.4.12. The graph of $-\ln(x-1)$ will intersect the x-axis at 2 since $-\ln(2-1)=0$. The only graph with this property is (a).

4.4.38. We can expand $\ln(x\sqrt[3]{x^2+1})$ as

$$\ln(x)+\ln((x^2+1)^{1/3})=\ln(x)+\frac{1}{3}\ln(x^2+1).$$

4.4.62. Multiplying both sides by the denominator to clear the fraction, we have

$$50=10.5(1+12e^{-0.02x})$$
$$39.5=126e^{-0.02x}$$
$$\frac{39.5}{126}=e^{-0.02x}$$

and taking the natural logarithm of both sides yields

$$\ln(\frac{39.5}{126})=-0.02x$$

so,

$$x=-50\ln(\frac{39.5}{126}).$$

4.4.74. (a) If $t=0$ in 1980 and t represents years, then the population in 2000 will be the value of the model at $t=20$.

$$P=2734.07e^{0.0210(20)}$$

which is about 4,161,149 people.

(b) To determine the year in which Houston will have a population of 6 million, we solve

$$6000=2734.07e^{0.0210t}$$

for t, recalling that the model gives population in thousands. Solving, we obtain

$$\frac{6000}{2734.07}=e^{0.0210t}$$
$$\ln(\frac{6000}{2734.07})=0.0210t$$

so,

$$t = \frac{1}{0.0210}\ln(\frac{6000}{2734.07})$$

which is about 37.4. So, Houston will have a population of 6 million in the year $1980 + 37 = 2017$.

4.5.20. To find the derivative of y we first simplify the expression as

$$y = \ln((\frac{x+1}{x-1})^{1/2}) = \frac{1}{2}(\ln(x+1) - \ln(x-1)).$$

Then, the derivative is

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}(\frac{1}{x+1}\frac{d}{dx}[x+1] - \frac{1}{x-1}\frac{d}{dx}[x-1]) \\ &= \frac{1}{2}(\frac{1}{x+1} - \frac{1}{x-1}).\end{aligned}$$

4.5.42. To find the derivative of y we first rewrite the expression as

$$y = 6^{5x} = e^{5x\ln(6)}$$

so the derivative is

$$y' = e^{5x\ln(6)}\frac{d}{dx}[5x\ln(6)] = 5\ln(6)6^{5x}.$$

4.5.46. To find the derivative of y we first rewrite the expression as

$$y = x3^{x+1} = xe^{(x+1)\ln(3)}.$$

Then, using the product rule we have

$$\begin{aligned}y' = \frac{d}{dx}[x]\,e^{(x+1)\ln(3)} + x\frac{d}{dx}\left[e^{(x+1)\ln(3)}\right] &= 3^{x+1} + x3^{x+1}\frac{d}{dx}[(x+1)\ln(3)] \\ &= 3^{x+1}(1+\ln(3)x).\end{aligned}$$

4.5.48. To find the equation of a line in the form $y = mx + b$ we need to compute the slope m and the y-intercept b. The slope is the value of the derivative at the given point. In this case, we have

$$y' = \frac{\frac{d}{dx}[\ln(x)]\,x - \ln(x)\frac{d}{dx}[x]}{x^2} = \frac{1-\ln(x)}{x^2}$$

using the quotient rule, and evaluating at the given point yields $y'(e) = 0$. Hence, the line is horizontal.

This horizontal line passes through the point $(e, \frac{1}{e})$ so it passes through the y-axis at $y = \frac{1}{e}$. Therefore, the equation of the tangent line is

$$y = \frac{1}{e}.$$

Problem. Find the derivative of $y = (1-x)^{\ln(x)}$.

We first rewrite this function as

$$y = e^{\ln(x)\ln(1-x)}$$

and differentiate as

$$\frac{dy}{dx} = e^{\ln(x)\ln(1-x)} \frac{d}{dx}\left[\ln(x)\ln(1-x)\right]$$

$$= (1-x)^{\ln(x)} \left(\frac{d}{dx}\left[\ln(x)\right]\ln(1-x) + \ln(x)\frac{d}{dx}\left[\ln(1-x)\right] \right)$$

$$= (1-x)^{\ln(x)} (\frac{1}{x}\ln(1-x) + \ln(x)\frac{1}{1-x}\frac{d}{dx}\left[1-x\right])$$

$$= (1-x)^{\ln(x)} (\frac{1}{x}\ln(1-x) - \ln(x)\frac{1}{1-x}).$$

4.6.4. We have two points and two unknown parameters in the curve $y = Ce^{kt}$, so we must plug in the points and solve for the parameters C and k. The first point yields

$$2 = Ce^{k(0)}$$

so $C = 2$. Using this, the second point yields

$$1 = 2e^{k(5)}$$

so

$$\frac{1}{2} = e^{5k}$$

and

$$5k = \ln(\frac{1}{2})$$

so

$$k = \frac{1}{5}\ln(\frac{1}{2})$$

Therefore, the exponential function passing through the given points is $y = 2(\frac{1}{2})^{t/5}$.

4.6.8. From the discussion in the chapter, we know that $y(t)$ has the form $Ce^{-\frac{2}{3}t}$ for some constant C. To find C we use the information given that $y(0) = 20$. Since $y(0) = C$, the equation for y is

$$y(t) = 20e^{-\frac{2}{3}t}.$$

This function represents exponential decay because the exponent is negative. We can check our answer by differentiating, and we obtain

$$y' = 20e^{-\frac{2}{3}t}\frac{d}{dx}\left[-\frac{2}{3}t\right] = -\frac{2}{3}(20e^{-\frac{2}{3}t}) = -\frac{2}{3}y(t)$$

as desired.

4.6.24. Although the question doesn't explicitly ask for a formula, it will be easier to answer the questions if we determine the equation for the exponential model. Let t be number of years with $t = 0$ corresponding to 1960. Let $E(t)$ be enrollment in millions of students at time t. Then, the exponential model has the form $E(t) = Ce^{kt}$ where we must determine C and k from the information given. Since $E(0) = 2.3$, we have $C = 2.3$. Using this, we can determine k from the other data point $E(40) = 12$, as

$$12 = 2.3e^{k(40)}$$

so

$$\frac{12}{2.3} = e^{40k}$$

and

$$\ln(\frac{12}{2.3}) = 40k$$

so

$$k = \frac{1}{40}\ln(\frac{12}{2.3}).$$

Therefore, the model is

$$E(t) = 2.3(\frac{12}{2.3})^{t/40}.$$

(a) The total enrollment in 1970 is $E(10)$ which is about 3.476 million. The total enrollment in 1980 is $E(20)$ which is about 5.254 million. The total enrollment in 1990 is $E(30)$ which is about 7.940 million.

(b) To find the doubling time from $t = 40$ when $E(t) = 12$, we solve

$$24 = 2.3(\frac{12}{2.3})^{t/40}$$

for t. We obtain

$$\frac{24}{2.3} = e^{(t/40)\ln(12/2.3)}$$

and taking the natural logarithm on both sides, we have

$$\ln(\frac{24}{2.3}) = (t/40)\ln(12/2.3)$$

which gives

$$t = 40\frac{\ln(24) - \ln(2.3)}{\ln(12) - \ln(2.3)}$$

which is about 56.7832. This occurs about 16 years after 2000 in the year 2016.

(c) The annual percentage increase can be seen by writing the model as $E(t) = 2.3\left((\frac{12}{2.3})^{1/40}\right)^t$. Then, the annual percentage increase is base of the exponential function $\left((\frac{12}{2.3})^{1/40}\right)^t$ minus 1. This follows because the base of the exponential function is the number that we multiply 2.3 by each year to compute the new enrollment. Hence, the annual percentage increase is

$$(\frac{12}{2.3})^{1/40} - 1$$

which is about 0.04216. Therefore, enrollments increase about 4.22% each year.