Homework 4

5.4.20. To evaluate $\int_2^7 3v\,dv$ we first find the antiderivative (also known as the indefinite integral)

$$\int 3v\,dv = \frac{3}{2}v^2 + C.$$

Then by the Fundamental Theorem of Calculus, we have

$$\int_2^7 3v\,dv = \left[\frac{3}{2}v^2\right]_2^7 = \left(\frac{3}{2}(7^2) - \frac{3}{2}(2^2)\right) = \frac{3}{2}(45).$$

5.4.30. To evaluate the definite integral $\int_0^1 \frac{x-\sqrt{x}}{3}\,dx$ we first find the antiderivative

$$\int \frac{x-\sqrt{x}}{3}\,dx = \frac{1}{3}\left(\int x\,dx - \int x^{1/2}\,dx\right) = \frac{1}{3}\left(\frac{x^2}{2} - \frac{2}{3}x^{3/2}\right) + C$$
$$= \frac{1}{6}x^2 - \frac{2}{9}x^{3/2} + C.$$

Then by the Fundamental Theorem of Calculus we have

$$\int_0^1 \frac{x-\sqrt{x}}{3}\,dx = \left[\frac{1}{6}x^2 - \frac{2}{9}x^{3/2}\right]_0^1 = \left(\frac{1}{6} - \frac{2}{9}\right) - (0-0)$$

which is $-\frac{1}{18}$.

5.4.42. To evaluate the definite integral $\int_0^1 \frac{e^{2x}}{e^{2x}+1}\,dx$ we first find the antiderivative

$$\int \frac{e^{2x}}{e^{2x}+1}\,dx.$$

Letting $u = e^{2x}+1$ we have $du = 2e^{2x}\,dx$ so $e^{2x}\,dx = \frac{1}{2}\,du$. We rewrite the indefinite integral as

$$\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\left(\ln|u| + C\right) = \frac{1}{2}\ln|e^{2x}+1| + C = \frac{1}{2}\ln(e^{2x}+1) + C$$

where we can drop the absolute value since $e^{2x}+1$ is always positive. Then by the Fundamental Theorem of Calculus we have

$$\int_0^1 \frac{e^{2x}}{e^{2x}+1}\,dx = \left[\frac{1}{2}\ln(e^{2x}+1)\right]_0^1 = \left(\frac{1}{2}\ln(e^2+1)\right) - \left(\frac{1}{2}\ln(e^0+1)\right)$$

$$= \frac{1}{2}(e^2 + 1) - \frac{1}{2}\ln 2.$$

5.4.46. The integrand $4 - |x|$ is equal to $4 - x$ on the interval $x = 0$ to $x = 4$ and the integrand $4 - |x|$ is equal to $4 - (-x) = 4 + x$ on the interval $x = -4$ to $x = 0$. Therefore, we must split the definite integral into two pieces.

$$\int_{-4}^{4} 4 - |x|\,dx = \int_{-4}^{0} 4 - |x|\,dx + \int_{0}^{4} 4 - |x|\,dx = \int_{-4}^{0} 4 + x\,dx + \int_{0}^{4} 4 - x\,dx$$

Then, we can proceed as usual. We obtain

$$\int_{-4}^{0} 4 + x\,dx + \int_{0}^{4} 4 - x\,dx = \left[4x + \frac{1}{2}x^2\right]_{-4}^{0} + \left[4x - \frac{1}{2}x^2\right]_{0}^{4}$$

$$(0 - (4(-4) + 8)) + ((16 - 8) - 0) = 8 + 8 = 16.$$

5.4.74. Observe that $f(x) = x^3$ is an odd function since

$$f(-x) = (-x)^3 = (-1)^3 x^3 = -(x^3) = -f(x).$$

This means that $\int_{-a}^{a} f(x)\,dx = 0$ for any a, including $a = 2$ so integral (b) is 0. Since we are given that $\int_0^2 x^3\,dx = 4$, we must then have that

$$0 = \int_{-2}^{2} x^3\,dx = \int_{-2}^{0} x^3\,dx + \int_{0}^{2} x^3\,dx = \int_{-2}^{0} x^3\,dx + 4$$

so integral (a) is

$$\int_{-2}^{0} x^3\,dx = 0 - 4 = -4.$$

For part (c), we have

$$\int_{0}^{2} 3x^3\,dx = 3\int_{0}^{2} x^3\,dx = 3 \cdot 4 = 12$$

by the linearity of integrals.

5.5.4. The region covers $x = 0$ to $x = 1$ and $f(x) \geq g(x)$. Hence, the area of the region is given by

$$\int_0^1 x^2 - x^3\,dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \left(\frac{1}{3} - \frac{1}{4}\right) - (0 - 0) = \frac{1}{12}.$$

5.5.8. The region covers $x = 1$ to $x = 2$ because the graphs intersect at x satisfying

$$-x + 3 = 2x^{-1}$$

and solving for x gives

$$-x^2 + 3x - 2 = 0$$

which we can write as

$$(-x + 1)(x - 2) = 0$$

so $x = 1$ or $x = 2$. Since $f(3/2) = 3/2 > 4/3 = g(3/2)$, we have that $f(x) \geq g(x)$ in the region. Therefore, the area of the region is given by

$$\int_1^2 -x+3-(2x^{-1})\,dx = \left[-\frac{x^2}{2} + 3x - 2\ln|x|\right]_1^2 = (-2 + 6 - 2\ln 2) - \left(-\frac{1}{2} + 3 - 2\ln 1\right)$$

$$= 1 + \frac{1}{2} - 2\ln 2 = \frac{3}{2} - 2\ln 2.$$

5.5.28. We begin by finding the intersection points of these functions by solving

$$y(2 - y) = -y$$

for y. We obtain

$$-y^2 + 2y + y = 0$$

which is

$$-y(y - 3) = 0$$

so $y = 3$ or $y = 0$. Also, we have $f(1) = 1 > -1 = g(1)$ so $f(y) \geq g(y)$ for all y between 0 and 3. Therefore, the area of the region bounded by these curves is

$$\int_0^3 y(2-y)-(-y)\,dy = \int_0^3 -y^2+3y\,dy = \left[-\frac{y^3}{3} + \frac{3y^2}{2}\right]_0^3 = \left(-9 + \frac{27}{2}\right)-(0+0) = \frac{9}{2}.$$