Homework 5

5.7.10. To find the points of intersection we solve $x = e^{x-1}$ for x. We have $1 = e^{1-1}$ so $x = 1$ is a solution. Since e^{x-1} is an exponential function that grows faster than the linear function $y = x$, it is the only solution. Since $0 < e^{0-1}$, we have that $y = x$ is less than $y = e^{x-1}$ for all x in the interval $[0, 1]$. By the washer method, the volume of the solid of revolution is

$$\pi \int_0^1 (e^{x-1})^2 - x^2\, dx = \pi \int_0^1 e^{2x-2}\, dx - \pi \int_0^1 x^2\, dx$$

To find this definite integral, we begin by finding the indefinite integrals. Using the substitution $u = 2x - 2$ we have $du = 2\, dx$ so $dx = \frac{1}{2}\, du$ and

$$\int e^{2x-2}\, dx = \frac{1}{2} \int e^u du = \frac{1}{2} e^u + C = \frac{1}{2} e^{2x-2} + C.$$

By the power rule, we have $\int x^2\, dx = \frac{x^3}{3}$. Hence, the volume of the solid of revolution is

$$\pi \left[\frac{1}{2} e^{2x-2} \right]_0^1 - \pi \left[\frac{x^3}{3} \right]_0^1 = \pi \left(\frac{1}{2}(1 - e^{-2}) - \frac{1}{3} \right) = \pi \left(\frac{1}{6} - \frac{1}{2e^2} \right).$$

This is a positive number, which is consistent with the fact that it represents a volume.

5.7.32. Using the disk method, we obtain the volume of the solid of revolution as

$$\pi \int_0^4 \left(\sqrt{\frac{x}{2}} + 1 \right)^2 dx = \pi \int_0^4 \frac{x}{2} + 2\sqrt{\frac{x}{2}} + 1\, dx$$

$$= \pi \left(\frac{1}{2} \int_0^4 x\, dx + \frac{2}{2^{1/2}} \int_0^4 x^{1/2}\, dx + \int_0^4 1\, dx \right)$$

$$= \pi \left(\frac{1}{2} \left[\frac{x^2}{2} \right]_0^4 + 2^{1/2} \left[\frac{x^{3/2}}{3/2} \right]_0^4 + [x]_0^4 \right)$$

$$= \pi \left(4 + \frac{2\sqrt{2}}{3} 4^{3/2} + 4 \right) = \pi \left(\frac{16\sqrt{2}}{3} + 8 \right)$$

cubic inches.

6.1.38. In this integral, our goal is to simplify the base of $(y+1)^{1/3}$ so that we can apply the power rule. Letting $u = y+1$ we obtain $du = dy$. We can find y^2 by first solving for $y = u-1$ and then squaring, so $y^2 = (u-1)^2 = u^2 - 2u + 1$. Substituting these, we get

$$\int y^2(y+1)^{1/3}\,dy = \int (u^2-2u+1)u^{1/3}\,du = \int u^{7/3} - 2u^{4/3} + u^{1/3}\,du$$

$$= \int u^{7/3}\,du - 2\int u^{4/3}\,du + \int u^{1/3}\,du.$$

Now, we can apply the power rule to each term:

$$= \frac{u^{10/3}}{10/3} - 2\frac{u^{7/3}}{7/3} + \frac{u^{4/3}}{4/3}$$

and substituting the y expressions back in for u, we have

$$= \frac{3}{10}(y+1)^{10/3} - \frac{6}{7}(y+1)^{7/3} + \frac{3}{4}(y+1)^{4/3}.$$

6.1.46. Letting $u = 1-x$ we have $du = -dx$ so $dx = -du$ and $x^2 = (1-u)^2 = u^2 - 2u + 1$. The limits of substitution with respect to u become $(1-0) = 1$ and $(1-\frac{1}{2}) = \frac{1}{2}$, respectively. Substituting, we obtain

$$\int_0^{1/2} x^2(1-x)^3\,dx = -\int_1^{1/2}(u^2-2u+1)u^3\,du = -\int_1^{1/2} u^5 - 2u^4 + u^3\,du.$$

We swap the limits of integration to get rid of the sign:

$$= \int_{1/2}^{1} u^5 - 2u^4 + u^3\,du.$$

$$= \left[\frac{u^6}{6} - 2\frac{u^5}{5} + \frac{u^4}{4}\right]_{1/2}^{1} = \left(\frac{1}{6} - \frac{2}{5} + \frac{1}{4}\right) - \left(\frac{1}{6\cdot 2^6} - 2\frac{1}{5\cdot 2^5} + \frac{1}{4\cdot 2^4}\right)$$

This turns out to be $\frac{7}{640}$.

6.1.58. By the definition of the definite integral, we have that the area under the curve of $\frac{1}{1+x^{1/2}}$ is

$$\int_0^4 (1+x^{1/2})^{-1}\,dx.$$

Our goal is to simplify the denominator so that we can apply the "log rule" for $\int \frac{1}{u}\,du$. Letting $u = 1 + x^{1/2}$ we obtain $du = \frac{1}{2}x^{-1/2}\,dx$. Since no $x^{-1/2}$ term appears in the integrand, we will have to rewrite this in terms of u before we can complete the substitution. Notice that $u - 1 = x^{1/2}$ so we can solve for dx in terms of u and du as

$$dx = 2x^{1/2}\,du = 2(u-1)\,du.$$

The limits of integration become $1+(0)^{1/2} = 1$ and $1+(4)^{1/2} = 3$, respectively. Now we have all the pieces to change the variable of integration from x to u:

$$\int_1^3 2u^{-1}(u-1)\,du = 2\int_1^3 1 - u^{-1}\,du = 2\left[u - \ln|u|\right]_1^3$$
$$= 2\left((3-\ln 3) - (1-\ln 1)\right) = 4 - 2\ln 3.$$

6.1.62. To find the average of $f(x) - g(x)$ on the interval $[0,2]$, we use the formula

$$\frac{1}{2-0}\int_0^2 f(x) - g(x)\,dx = \frac{1}{2}\int_0^2 x(4x+1)^{1/2} - 2x^{3/2}\,dx.$$
$$= \frac{1}{2}\int_0^2 x(4x+1)^{1/2}\,dx - \int_0^2 x^{3/2}\,dx.$$

We must simplify the base of $(4x+1)^{1/2}$ by substituting $u = 4x+1$ so $du = 4\,dx$ and $dx = \frac{1}{4}\,du$ and $x = \frac{1}{4}(u-1)$. The limits of integration become $4(0)+1 = 1$ and $4(2)+1 = 9$, respectively. Hence, we obtain

$$= \frac{1}{2}\int_1^9 \frac{1}{16}(u-1)u^{1/2}\,du - \int_0^2 x^{3/2}\,dx$$
$$= \frac{1}{32}\int_1^9 u^{3/2} - u^{1/2}\,du - \int_0^2 x^{3/2}\,dx$$

$$= \frac{1}{32}\left[\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right]_1^9 - \left[\frac{x^{5/2}}{5/2}\right]_0^2$$

$$= \frac{1}{32}\left(\left(\frac{2}{5}(9)^{5/2} - \frac{2}{3}(9)^{3/2}\right) - \left(\frac{2}{5}(1)^{5/2} - \frac{2}{3}(1)^{3/2}\right)\right) - \left(\frac{2}{5}(2)^{5/2} - \frac{2}{5}(0)^{5/2}\right)$$

$$= \frac{149}{60} - \frac{8\sqrt{2}}{5}.$$

6.2.6. We apply integration by parts letting $dv = dx$ and $u = \ln x^2$ so $v = x$ and $du = \frac{2x}{x^2}\,dx = \frac{2}{x}\,dx$, so

$$\int \ln x^2\,dx = uv - \int v\,du = x\ln x^2 - \int x\frac{2}{x}\,dx$$

$$= x\ln x^2 - 2\int 1\,dx = x\ln x^2 - 2x + C.$$

6.2.14. We apply integration by parts letting $dv = e^{-x}\,dx$ and $u = x$ so $v = -e^{-x}$ and $du = dx$. Then,

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C.$$

6.2.18. Recognizing that $\frac{1}{x}$ is the derivative of $\ln x$ which appears in the integrand, we choose to use substitution. Let $u = \ln x$ so $du = \frac{1}{x}\,dx$ and we have

$$\int \frac{1}{x(\ln x)^3}\,dx = \int u^{-3}\,du = \frac{u^{-2}}{-2} = -\frac{1}{2}(\ln x)^{-2}.$$

6.2.26. Working toward the power rule, we use substitution with $u = 2 + 3x$ so $du = 3\,dx$ and $dx = \frac{1}{3}\,du$ and $x = \frac{1}{3}(u-2)$. Then,

$$\int \frac{x}{\sqrt{2+3x}}\,dx = \int \frac{1}{9}(u-2)u^{-1/2}\,du = \frac{1}{9}\int u^{1/2} - 2u^{-1/2}\,du$$

$$= \frac{1}{9}\left(\frac{u^{3/2}}{3/2} - 2\frac{u^{1/2}}{1/2}\right) = \frac{2}{27}u^{3/2} - \frac{4}{9}u^{1/2}.$$

$$= \frac{2}{27}(2+3x)^{3/2} - \frac{4}{9}(2+3x)^{1/2}.$$