Homework 6

6.2.30. We apply integration by parts letting $dv = e^{-x}\,dx$ and $u = x^2$ so $v = -e^{-x}$ and $du = 2x\,dx$. Then,

$$\int_0^2 x^2 e^{-x}\,dx = \left[-e^{-x}x^2 - 2\int -e^{-x}x\,dx\right]_0^2 = \left[-e^{-x}x^2 + 2\int e^{-x}x\,dx\right]_0^2.$$

To simplify $\int e^{-x}x\,dx$ we must apply integration by parts again with $dv = e^{-x}\,dx$ and $u = x$ so $v = -e^{-x}$ and $du = dx$. Then, we have

$$\left[-e^{-x}x^2 + 2\left(-xe^{-x} - \int -e^{-x}\,dx\right)\right]_0^2 = \left[-e^{-x}x^2 + 2\left(-xe^{-x} - e^{-x}\right)\right]_0^2$$

$$= \left(-4e^{-2} - 4e^{-2} - 2e^{-2}\right) - \left(0 + 0 - 2e^0\right) = 2 - 10e^{-2}.$$

6.2.34. We apply substitution to simplify the integrand, letting $u = 1+2x$ so $du = 2\,dx$ and $dx = \frac{1}{2}\,du$. When $x = 0$, we have $u = 1 + 2(0) = 1$ and when $x = 1$, we have $u = 1 + 2(1) = 3$. Hence, we can rewrite the integral as

$$\frac{1}{2}\int_1^3 \ln(u)\,du.$$

To simplify notation, we rewrite this with $u = x$ so that we can apply integration by parts:

$$\frac{1}{2}\int_1^3 \ln(x)\,dx.$$

Let $dv = dx$ and $u = \ln(x)$ so $v = x$ and $du = \frac{1}{x}\,dx$. Then we have

$$\frac{1}{2}\left[x\ln(x) - \int x\frac{1}{x}\,dx\right]_1^3 = \frac{1}{2}\left[x\ln(x) - x\right]_1^3$$

$$= \frac{1}{2}\left((3\ln 3 - 3) - (1\ln 1 - 1)\right) = \frac{3}{2}\ln 3 - 1.$$

6.3.18. The integrand is a proper rational function, so we can find its partial fraction decomposition as

$$\frac{3}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3}.$$

The basic equation is

$$3 = A(x-3) + B(x)$$

which we can solve by plugging in $x = 3$ to get $3B = 3$ so $B = 1$ and $x = 0$ to get $-3A = 3$ so $A = -1$. Hence our integral is

$$\int \frac{1}{x-3} - \frac{1}{x}\,dx = \ln|x-3| - \ln|x| + C.$$

Here, we are using that $\int \frac{1}{x+a}\,dx = \ln|x+a| + C$ for any constant a. This can be proved by substitution because when $u = x+a$, we have $du = dx$. Notice that if we integrate $\int \frac{1}{-x+a}\,dx$ then we pick up a sign, and if we integrate $\int \frac{1}{bx+a}\,dx$ then we will pick up a coefficient of $\frac{1}{b}$.

6.3.26. The integrand is a proper rational function and the denominator factors as $x(x^2-1) = x(x-1)(x+1)$, so we can find its partial fraction decomposition as

$$\frac{3x^2-7x-2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}.$$

The basic equation is

$$3x^2 - 7x - 2 = A(x-1)(x+1) + B(x)(x+1) + C(x)(x-1)$$

which we can solve by plugging in $x = 1$ to get $2B = 3 - 7 - 2 = -6$ so $B = -3$ and $x = -1$ to get $2C = 3 + 7 - 2 = 8$ so $C = 4$ and $x = 0$ to get $-A = -2$ so $A = 2$. Hence our integral is

$$\int \frac{2}{x} - \frac{3}{x-1} + \frac{4}{x+1}\,dx = 2\ln|x| - 3\ln|x-1| + 4\ln|x+1| + C.$$

6.3.30. The integrand is not proper, so we must first perform long division on the rational function. We obtain

$$\frac{x^4}{x^3 - 3x^2 + 3x - 1} = x + 3 + \frac{6x^2 - 8x + 3}{(x-1)^3}.$$

We find the partial fraction decomposition of the remainder as

$$\frac{6x^2 - 8x + 3}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

and the basic equation is

$$6x^2 - 8x + 3 = A(x-1)^2 + B(x-1) + C = Ax^2 + (B-2A)x + (C-B+A).$$

Comparing the coefficients of x^2 on both sides, we have $A = 6$. Comparing the coefficients of x on both sides, we have $B - 2A = -8$ so $B = 4$. Comparing the coefficients of 1 on both sides, we have $C - B + A = 3$ so $C = 1$. Hence, our integral becomes

$$\int x + 3 + \frac{6}{x-1} + 4(x-1)^{-2} + (x-1)^{-3}$$

$$= \frac{x^2}{2} + 3x + 6\ln|x-1| - 4(x-1)^{-1} - \frac{1}{2}(x-1)^{-2} + C$$

$$= \frac{x^2}{2} + 3x + 6\ln|x-1| - \frac{8x-7}{2(x-1)^2} + C.$$

6.3.38. The integrand is not proper, so we must first perform long division on the rational function. We obtain

$$\frac{x^3-1}{x^2-4} = x + \frac{4x-1}{x^2-4}.$$

We factor the denominator of the remainder as a difference of squares $x^2 - 4 = (x-2)(x+2)$ and find the partial fraction decomposition of the remainder as

$$\frac{4x-1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}.$$

The basic equation is

$$4x - 1 = A(x+2) + B(x-2)$$

and we plug in $x = -2$ to obtain $-4B = -9$ so $B = \frac{9}{4}$ as well as $x = 2$ to obtain $4A = 7$ so $A = \frac{7}{4}$. Hence, our integral becomes

$$\int_0^1 x + \frac{7}{4}\frac{1}{x-2} + \frac{9}{4}\frac{1}{x+2}\,dx = \left[\frac{x^2}{2} + \frac{7}{4}\ln|x-2| + \frac{9}{4}\ln|x+2|\right]_0^1.$$

$$= \frac{1}{2} + \frac{9}{4}\ln 3 - 4\ln 2.$$

6.3.44. We begin by determining where the curve meets the x-axis by solving $\frac{x^2+2x-1}{x^2-4} = 0$ for x. In order for this fraction to be zero, we must have that the numerator is zero (and the denominator is non-zero). Hence, we solve $x^2 + 2x - 1 = 0$ using the quadratic formula, so $x = \frac{-2\pm\sqrt{4-4(-1)}}{2}$. The positive root occurs at $\sqrt{2} - 1$. (This turns out to be about 0.41.) Notice that the denominator $x^2 - 4$ does not vanish at this root, so the function is well-defined at $x = \sqrt{2} - 1$.

By definition, the area under this curve is

$$\int_{-1}^{\sqrt{2}-1} \frac{x^2 + 2x - 1}{x^2 - 4}\, dx$$

which we need partial fractions to solve. The integrand is not proper so we must first perform long division to obtain

$$\frac{x^2 + 2x - 1}{x^2 - 4} = 1 + \frac{2x + 3}{x^2 - 4}.$$

The denominator of the remainder factors as a difference of squares $(x^2 - 4) = (x - 2)(x + 2)$ and so the partial fraction decomposition is

$$\frac{2x + 3}{x^2 - 4} = \frac{A}{x - 2} + \frac{B}{x + 2}$$

with basic equation

$$2x + 3 = A(x + 2) + B(x - 2).$$

Plugging in $x = -2$ we get $-4B = -1$ so $B = \frac{1}{4}$ and $x = 2$ gives $4A = 7$ so $A = \frac{7}{4}$. Hence, our integral becomes

$$\int_{-1}^{\sqrt{2}-1} 1 + \frac{7}{4}\frac{1}{x - 2} + \frac{1}{4}\frac{1}{x + 2}\, dx$$

$$= \left[x + \frac{7}{4}\ln|x - 2| + \frac{1}{4}\ln|x + 2|\right]_{-1}^{\sqrt{2}-1}$$

$$= \sqrt{2} + \frac{7}{4}\left(\ln|\sqrt{2} - 3| - \ln 3\right) + \frac{1}{4}\ln|\sqrt{2} + 1|.$$