Homework 8

9.2.20. We observe that $f(x)$ is nonnegative and continuous on $[a,b]$ if $k \geq 0$; in fact, it is a constant function. Therefore, in order for $f(x)$ to be a PDF on $[a,b]$, we must have that $\int_a^b f(x)\,dx = 1$. Solving for k, we have

$$1 = \int_a^b \frac{k}{b-a}\,dx = \frac{k}{b-a}\int_a^b 1\,dx = \frac{k}{b-a}(b-a) = k$$

so $k = 1$. (This PDF is called the *uniform probability density function.*)

9.2.24. You can sketch this using software such as Sage. To find the probability that x lies between a and b, we find

$$\int_a^b \frac{5}{4(x+1)^2}\,dx = \frac{5}{4}\left[\frac{(x+1)^{-1}}{-1}\right]_a^b = -\frac{5}{4}\left(\frac{1}{b+1} - \frac{1}{a+1}\right).$$

Hence, we have:

(a) $\frac{5}{4}(1 - \frac{1}{3}) = \frac{5}{4}\frac{2}{3} = \frac{5}{6}$.
(b) $\frac{5}{4}(\frac{1}{3} - \frac{1}{5}) = \frac{5}{4}\frac{2}{15} = \frac{1}{6}$.
(c) $\frac{5}{4}(\frac{1}{2} - \frac{1}{4}) = \frac{5}{4}\frac{1}{4} = \frac{5}{16}$.
(d) $\frac{5}{4}(1 - \frac{1}{4}) = \frac{5}{4}\frac{3}{4} = \frac{15}{16}$.

9.2.28. To find the probability that x lies between a and b, we find

$$\int_a^b \frac{5}{324}t\sqrt{9-t}\,dx.$$

We use the substitution $u = 9 - t$, so $du = -dt$ and $t = 9 - u$. Then, the integral becomes

$$-\frac{5}{324}\int_{9-a}^{9-b}(9-u)u^{1/2}\,du = -\frac{5}{324}\int_{9-a}^{9-b} 9u^{1/2} - u^{3/2}\,du$$

$$= -\frac{5}{324}\left[9\frac{u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2}\right]_{9-a}^{9-b}$$

$$= -\frac{5}{324}\left[\left(6(9-b)^{3/2} - \frac{2}{5}(9-b)^{5/2}\right) - \left(6(9-a)^{3/2} - \frac{2}{5}(9-a)^{5/2}\right)\right]$$

Hence, we have:

(a) Using $a = 0$ and $b = 3$, we obtain

$$= -\frac{5}{324}\left[\left(6(9-3)^{3/2} - \frac{2}{5}(9-3)^{5/2}\right) - \left(6(9-0)^{3/2} - \frac{2}{5}(9-0)^{5/2}\right)\right]$$

$$= -\frac{5}{324}\left[\left(\frac{3}{5}6^{5/2}\right) - \left(\frac{4}{15}3^{5}\right)\right].$$

This is about 18%.

(b) Using $a = 4$ and $b = 8$, we obtain

$$= -\frac{5}{324}\left[\left(6(9-8)^{3/2} - \frac{2}{5}(9-8)^{5/2}\right) - \left(6(9-4)^{3/2} - \frac{2}{5}(9-4)^{5/2}\right)\right]$$

$$= -\frac{5}{324}\left[\left(\frac{28}{5}\right) - \left(6(5)^{3/2} - \frac{2}{5}(5)^{5/2}\right)\right]$$

This is about 60%.

9.3.4. You can sketch this PDF using software such as Sage. To find the mean μ of $f(x) = \frac{4}{3x^2}$, we integrate

$$\int_1^4 f(x)x\,dx = \int_1^4 \frac{4x}{3x^2}\,dx$$

and using the substitution $u = 3x^2$, we obtain

$$\frac{2}{3}\int_3^{48} \frac{1}{u}\,du = \frac{2}{3}(\ln 48 - \ln 3) = \frac{2}{3}(\ln 3 \cdot 4^2 - \ln 3) = \frac{4}{3}\ln 4.$$

This is about 1.85. To obtain the variance $V(x)$, we integrate

$$\int_1^4 f(x)x^2\,dx - \mu^2 = \int_1^4 \frac{4x^2}{3x^2}\,dx - \mu^2 = \frac{4}{3}\int_1^4 1\,dx - \mu^2$$

$$= \frac{4}{3}(3) - (\frac{4}{3}\ln 4)^2 = 4 - \frac{16}{9}(\ln 4)^2 = (2 - \frac{4}{3}\ln 4)(2 + \frac{4}{3}\ln 4).$$

This is about 0.58. Then, the standard deviation is $\sigma = \sqrt{V(x)}$. This is about 0.76.

9.3.12. To find the median, we solve

$$\int_0^m f(t)\,dt = \frac{1}{2}$$

for m. We integrate using the substitution $u = -2t/5$.

$$\frac{2}{5}\int_0^m e^{-2t/5}\,dt = -\int_0^{-2m/5} e^u\,du = 1 - e^{-2m/5}.$$

If we set this equal to $\frac{1}{2}$, we obtain

$$\begin{aligned} 1 - e^{-2m/5} &= \frac{1}{2} \\ \frac{1}{2} &= e^{-2m/5} \\ \ln\frac{1}{2} &= \frac{-2m}{5} \\ \frac{5}{2}\ln 2 &= m. \end{aligned}$$

9.3.32. The problem tells us that the service time follows an exponential PDF, and we are given that the mean is 3.5 minutes. Since the general formula for an exponential PDF is $f(t) = ae^{-at}$ with mean $1/a$, we have that $a = \frac{2}{7}$ so our PDF is

$$f(t) = \frac{2}{7}e^{-2t/7}.$$

To find the probability that t is within one standard deviation of the mean, we will have to find the standard deviation. The book assures us that this has a nice answer, so we might as well derive it for the general exponential PDF. The variance is given by

$$\int_0^\infty ax^2e^{-ax}\,dx - \mu^2$$

This is an improper integral, so we rewrite it as

$$\lim_{b\to\infty}\int_0^b ax^2e^{-ax}\,dx - \mu^2$$

Using the substitution $u = -ax$, we have

$$\lim_{b\to\infty} -\int_0^{-ab}\frac{u^2}{a^2}e^u\,du - \frac{1}{a^2}$$

$$= \frac{1}{a^2}\left(\lim_{b\to\infty} - \int_0^{-ab} u^2 e^u\, du - 1\right)$$

After two applications of integration by parts, we obtain

$$= \frac{1}{a^2}\left(\lim_{b\to\infty} - \left[(u^2 - 2u + 2)e^u\right]_0^{-ab} - 1\right)$$

Taking the limit, we have that $e^{-ab} \to 0$ as $b \to \infty$, so we are left with

$$= \frac{1}{a^2}(2-1)$$

$$= \frac{1}{a^2}.$$

Hence, the standard deviation is $\sqrt{\frac{1}{a^2}}$ which is the same as the mean for this PDF.

Now, the probability that t is within $\frac{7}{2}$ of the mean is given by

$$\int_{\mu-\frac{7}{2}}^{\mu+\frac{7}{2}} f(t)\, dt$$

$$= \int_0^7 \frac{2}{7} e^{-2t/7}\, dt.$$

Using the substitution $u = -2t/7$, we have

$$-\int_0^{-2} e^u\, du = \int_{-2}^0 e^u\, du = 1 - e^{-2}.$$

This is about 86%.

9.3.34. To set this problem up, we use a normal distribution with the given mean and standard deviation. This PDF is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} = \frac{1}{(10.5)\sqrt{2\pi}} e^{-(x-120)^2/2(10.5)^2}$$

To find the lowest score in the top 10% we would solve

$$\int_a^\infty \frac{1}{(10.5)\sqrt{2\pi}} e^{-(x-120)^2/2(10.5)^2} = \frac{1}{10}$$

for a. You can use numerical integration (on a computer) to determine that the solution a is about 133.