First, I apologize for how late this is. But at least it's something. And I still feel that it is useful to post the day before the exam, because I am including advice that will help you throughout your mathematical careers. I've taken a few review problems, and written up several "false solutions." Some of them may seem silly, and some of them may be only subtly wrong. But they are the types of mistakes you should be very careful not to make. Then, I have included proper solutions to them.

Here is some general advice that you should take to heart:

- A proof should be clear enough that anyone who reads the argument will have no choice but to agree with it. That means every single step should be completely obvious to the reader.

- Read over your own work after a few minutes, and try to reconstruct your argument. If you can't figure out what you're saying, no one else will be able to.

- If you are not sure whether or not you should prove something, prove it.

- Before you state anything, think about what you're claiming. Does it make sense? Make sure there are no obvious counter-examples. If there are, maybe you're remembering the theorem slightly wrong. What would make it right?

- Be careful!! The devil is in the details. Make sure you pay attention to them.

Good luck!!

Exercise 1. *Suppose s_n is a sequence which converges to a negative number s. Carefully prove that the terms of s_n must eventually become and stay negative.*

False Solution 1. *We prove by contradiction. Suppose the terms of s_n do not eventually become and stay negative. Then they must eventually become and stay positive. But it is impossible for a sequence of positive terms to converge to a negative number. Therefore, the terms of s_n must eventually become and stay negative.*

Explanation 1. *The problem with this solution is that the negation of "the terms of s_n eventually become and stay negative" is not "the terms of s_n eventually become and stay positive." It is "the terms of s_n do not eventually become and stay negative."*

Symbolically, "for all N, there exists an $n > N$ such that s_n is not negative." To translate once more, "no matter how far out in the sequence you go, there is always a later term that is not negative."

False Solution 2. *$s_n \to s$ means for any $\epsilon > 0$, there is an N such that for all $n > N$, $|s_n - s| < \epsilon$. Let $N > 1/\epsilon$. Then $s_n - s < 1/N$, so $s_n < 1/N + s$. And since $lim 1/N = 0$, we have $s_n < s < 0$.*

Explanation 2. *The idea is not bad here, but it is unclearly and wrongly worded. We do not select an N. We get to choose an ϵ in this context, because we know the statement is true for all ϵ. Once an ϵ is chosen, then we can try to determine what N is necessary. But in general, N should not be expressed in terms of just ϵ. It should also take into account the terms s_n, since the rate at which the s_n converge to s will determine how far out in the sequence one needs to go to be ϵ-close to our limit.*

Solution 1. *Let $\epsilon = |s|$. Then since $lim s_n = s$ means given ϵ, there is an N such that for all $n > N$, $|s_n - s| < \epsilon$. Then*

$$-\epsilon < s_n - s < \epsilon$$

$$s - \epsilon < s_n < s + \epsilon = 0$$

. Therefore, $s_n < 0$ for all $n > N$. That is, s_n eventually becomes and stays negative.

Exercise 2. *Find the interior of $\mathbb{Q}$.*

False Solution 3. *$\mathbb{Q}$ has no interior, because there are a bunch of gaps. Any ball around a point in $\mathbb{Q}$ will not be in $\mathbb{Q}$.*

Explanation 3. *This is very much the right intuition, but it is incomplete. It simply states what is true, but does not demonstrate it. How do we know that any ball around a point will not be contained in $\mathbb{Q}$. What does that have to do with anything?*

False Solution 4. *$\mathbb{Q}$ cannot have any interior, because it goes off to infinity.*

Explanation 4. *It is true that $\mathbb{Q}$ has infinitely many points, and that it goes off to infinity. But these have nothing to do with whether or not a set has interior. We need to appeal to the definition of an interior point to answer the question.*

False Solution 5. *Every point of $\mathbb{Q}$ is interior, because for any ball of radius r, there are rational points in the ball.*

Explanation 5. *This just confuses the definition of an interior point. It is true that interiorness has to do with balls around points. But the statement is that there exists a ball around a point which is completely contained in the set.*

Solution 2. *The interior of $\mathbb{Q}$ is empty. What it means for a point to be interior to a set is that there exists some positive radius $r > 0$ such that the ball of radius r about the point is completely contained in the set. So, given any $q \in \mathbb{Q}$, and any positive radius $r > 0$, we consider the ball $B_r(q) = \{x \in \mathbb{R} : |x - q| < r\} = (x - r, x + r)$. By the density of the irrationals, we know that there exists an irrational number z in $(x - r, x + r)$. This, is the precise thing we mean when we say that $\mathbb{Q}$ has "gaps." So the ball of radius r is not contained in $\mathbb{Q}$. But r was arbitrary, so q cannot be in interior point. But q was also arbitrary in $\mathbb{Q}$, so the interior of $\mathbb{Q}$ is empty.*

Exercise 3. *Consider the series $\sum a_n$ and $\sum b_n$ and suppose that the set $\{n \in \mathbb{N} : a_n \neq b_n\}$ is finite. Then the series both converge, or else they both diverge.*

False Solution 6. *Convergence is only about what happens at the end of the series, so you can mess up some terms at the beginning, and the convergence will be the same.*

Explanation 6. *This is exactly right, but totally incomplete. There is no explanation of why convergence is only about what happens at the end. There is no proof.*

False Solution 7. *A series converges if and only if it satisfies the Cauchy Criterion. That means for any $\epsilon > 0$, there is some N such that for any $n \geq m > N$, $|\sum_{k=n}^{m} a_k| < \epsilon$. So $\sum a_n$ satisfies the Cauchy Criterion if and only if $\sum b_n$ does.*

Explanation 7. *This was going along perfectly. But it is incomplete. There is no justification or proof of the fact that $\sum a_n$ and $\sum b_n$ either satisfy the Cauchy Criterion together or not. It is true, because the statement of the problem tells us that the series converge or diverge together. But the explanation is little more than rewriting the problem as a statement instead of a question.*

False Solution 8. $\sum_{n=0}^{\infty} a_n = lim_{k\to\infty} \sum_{n=0}^{k} a_n = lim_{k\to\infty} \sum_{n=0}^{k} b_n = \sum_{n=0}^{\infty} b_n$. *Therefore, they either both converge, or both diverge.*

Explanation 8. *While it is true that* $lim a_n = lim b_n$*, since they only differ in a finite number of terms, that does not make it the case that* $lim_{k\to\infty} \sum_{n=0}^{k} a_n = lim_{k\to\infty} \sum_{n=0}^{k} b_n$. *This is not true, for if it were, then* $\sum a_n$ *and* $\sum b_n$ *would actually converge to the same number.*

Solution 3. *A series converges if and only if it satisfies the Cauchy Criterion. That means for any* $\epsilon > 0$*, there is some* N *such that for any* $n \geq m > N$, $|\sum_{k=n}^{m} a_k| < \epsilon$. *But if* a_n *and* b_n *only differ in a finite number of terms, then we can take this* $N > sup\{n \in \mathbb{N} : a_n \neq b_n\}$. *Then, for all* $n \geq m > N$*, we actually have* $|\sum_{k=n}^{m} a_k| = |\sum_{k=n}^{m} b_k|$. *So together, this expression is either* $< \epsilon$*, or not.*

Exercise 4. *Let* S *be a bounded nonempty subset of* $\mathbb{R}$ *and suppose* $\sup S \notin S$. *Prove that there is a nondecreasing sequence* (s_n) *of points in* S *such that* $lim s_n = \sup S$.

False Solution 9. *S is bounded, so it has a convergent monotone subsequence. Since the sequence is nondecreasing, it must be getting larger, so it converges to* $\sup S$.

Explanation 9. *First, the theorem about the existence of a convergent subsequence is about bounded sequences, not bounded sets. To find a sequence, or subsequence in a set, you have to actually pick elements from the set. Second, there was a jump from a monotone sequence to a nondecreasing sequence. That is just wishful thinking. Third, even if the sequence were inreasing, that does not mean that it has to approach the* $\sup$ *of* S. *An increasing sequence* (s_n) *must approach the* $\sup\{s_n\}$. *But an increasing sequence in a set need not approach the* $\sup$ *of the set. Consider the sequence* $(1-1/n)$ *in* $[0, 5]$. *The sequence is increasing, converges, and gets nowhere near* 5.

False Solution 10. *We will construct the sequence "by hand." Let* $s = \sup S$. *S is nonempty, so there must be something in it. Call this point* s_1. *Now, since* $s = \sup S$, $s_1 \leq s$. *But we also know* $s_1 \in S, ands \notin S$. *So* $s_1 < s$. *Then, by the properties of* $\sup$*, since* $s_1 < s$*, there exists a point, call it* $s_2 \in S$*, with* $s_2 > s_1$. *But similarly,* $s_2 < s$. *So there exists* $s_3 \in S$*, with* $s_3 > s_2$*, and* $s_3 < s$. *In this manner, we can construct an entire sequence* (s_n) *which is increasing and converges to* s.

Explanation 10. *This is a really good idea. But the sequence constructed does not necessarily converge to* $\sup S$. *Again, just because the sequence is increasing does not mean that it is not bounded above by something below the* sup.

Solution 4. *Let* $s = \sup S$. *Then,* $s - 1$ *is not an upper bound for* S*, so by the property of* sup*, there must exist* $s_1 \in S$ *such that* $s_1 > (s - 1)$*, just like above. And again,* $s_1 \in S$*,* $s \notin S$ *says* $s_1 < s$*. So we have* $s - 1 < s_1 < s$*. Now,* $\max\{s - 1/2, s_1\}$ *is not an upper bound for* S*, so there exists* $s_2 \in S$ *such that* $\max\{s - 1/2, s_1\} < s_2 < s$*. Now,* $\max\{s - 1/4, s_2\}$ *is not an upper bound for* S*, so there exists* $s_3 \in S$ *such that* $\max\{s - 1/4, s_2\} < s_3 < s$*. If we proceed in this manner, then not only do we construct a sequence that is increasing, but we also guarantee that* $s - 1/n < s_n < s$*. So* $\lim s_n = s$.