Exercise 1. *Ross, 9.6. Let* $x_1 = 1$ *and* $x_{n+1} = 3x_n^2$, *for* $n \geq 1$.

(a) Show that if $a = lim(x_n)$, *then* $a = \frac{1}{3}$ *or* $a = 0$.

(b) Does $lim(x_n)$ *exist? Explain.*

(c) Discuss the apparent contradiction between parts (a) and (b).

Exercise 2. *Suppose* (p_n) *and* (q_n) *are two Cauchy sequences. Prove that* $(|p_n - q_n|)$ *converges.*

Exercise 3. *Ross, 9.10. (a) Show that if* $lim(s_n) = \infty$ *and* $k > 0$, *then* $lim(ks_n) = \infty$.

(b) Show that $lim(s_n) = \infty$ *if and only if* $lim(-s_n) = -\infty$.

(c) Show that if $lim(s_n) = \infty$ *and* $k < 0$, *then* $lim(ks_n) = -\infty$.

Solutions

Exercise 1.

Proof. (a) Suppose $a = lim(x_n)$ exists. Then $a = lim(x_{n+1}) = lim(3x_n^2) = 3a^2$. So solving the equation $3a^2 - a = a(3a-1) = 0$ yields $a = 0$ or $a = \frac{1}{3}$.

(b) No, $lim(x_n) = \infty$, because x_n increases without bound.

(c) Part (a) showed that [a real limit exists $\implies$ the limit is 0 or $\frac{1}{3}$]. But part (b) shows that the limit does not exist. The hypothesis of the implication is not satisfied, so the conclusion need not be true.

□

Exercise 2.

Proof. Since $(|p_n - q_n|)$ is a sequence of real numbers, it suffices to show that it is a Cauchy sequence. (By Theorem 10.11.)

Let $\epsilon > 0$.

Since (p_n) is Cauchy, there is an N_1 such that for all $n, m > N_1$, we have $|p_n - p_m| < \frac{\epsilon}{2}$.

Similarly, there is an N_2 such that for all $n, m > N_2$, we have $|q_n - q_m| < \frac{\epsilon}{2}$.

So take $N = max(N_1, N_2)$. Then for all $n, m > N$, we have $||p_n - q_n| - |p_m - q_m|| \leq |p_n - q_n - p_m + q_m|$ because $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

Then $|p_n - q_n - p_m + q_m| \leq |p_n - p_m| + |q_m - q_n|$ by the Triangle Inequality.

So $||p_n - q_n| - |p_m - q_m|| \leq |p_n - p_m| + |q_m - q_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, for all $n, m > N$.

Therefore, $(|p_n - q_n|)$ is Cauchy, and hence converges. □

Exercise 3.

Proof. (a) Let $M > 0$.

Since $lim(s_n) = \infty$, there is an N such that for all $n > N$, $s_n > \frac{M}{k}$.

Then, for all $n > N$, $ks_n > M$. Thus, $lim(s_n) = \infty$.

(b) Suppose $lim(s_n) = \infty$. Let $M < 0$.

Then $-M > 0$. There exists an N such that for all $n > N$, $s_n > -M$. Which implies $-s_n < M$.

Therefore, $lim(-s_n) = -\infty$.

Conversely, suppose $lim(-s_n) = -\infty$. Let $M > 0$.

Then $-M < 0$. There exists N such that for all $n > N$, $-s_n < -M$, which implies $s_n > M$.

(c) Suppose $lim(s_n) = \infty$, and $k < 0$. Let $M < 0$. Then $\frac{M}{k} > 0$.

There exists an N such that for all $n > N$, $s_n > \frac{M}{k}$, which implies $ks_n < M$.

Therefore, $lim(ks_n) = -\infty$. □