FINAL EXAM

This is the final exam for Math 25, Fall 2007. Please write your name clearly at the top of the exam. The exam has 100 points, and you have 120 minutes to complete this exam. You may not use any notes or books, nor any calculating or computing devices. Please give *as much justification as you can* for all of your solutions.

1. (15 points) (a) Criticize the following argument:

$$\lim_{n\to\infty} \frac{\sin n}{n} = (\lim_{n\to\infty} \sin n) \cdot (\lim_{n\to\infty} \frac{1}{n}) = (\lim_{n\to\infty} \sin n) \cdot 0 = 0.$$

(b) Give a valid argument showing that $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

The problem with the argument given in (a) is that $\sin n$ does not converge so we cannot apply the product theorem for limits. In this case, we can still give a valid argument because $\sin n$ is bounded, but note that the same proof as (a) would show that $\lim n = \lim \frac{n^2}{n}$ is 0, which is false.

To show that $\lim_{n\to\infty} \frac{\sin n}{n} = 0$, let $\epsilon > 0$ and choose $N = \frac{1}{\epsilon}$. Then, $n > N$ implies that

$$\epsilon > \frac{1}{n} \geq \frac{|\sin n|}{n} = |\frac{\sin n}{n}|$$

as desired.

2. (15 points) (a) Prove carefully that the open interval (a, b) is an open subset of $\mathbb{R}$.

(b) Prove directly that any union of open intervals is an open subset of $\mathbb{R}$.

(c) Give an example showing that an infinite intersection of open intervals need not be an open subset of $\mathbb{R}$.

Let $a < x < b$ so $r = \min(x - a, b - x) > 0$. Then, if $|y - x| < r$ we have

$$-r < y - x < r$$

so $y < r + x \leq (b - x) + x = b$ and $y > x - r \geq x - (x - a) = a$, proving (a).

Suppose $\mathcal{U}$ is some union of open intervals. Then for any $x \in \mathcal{U}$ there exists an open interval (a, b) containing x. By (a), there exists $r > 0$ such that $\{y : |y - x| < r\} \subset (a, b) \subset \mathcal{U}$, so $\mathcal{U}$ is open.

If we consider

$$\mathcal{U} = \bigcap_{n=1}^{\infty} (0, 1 + \frac{1}{n}) \bigcap (1 - \frac{1}{n}, 2)$$

we see that $\mathcal{U} = \{1\}$ which is not open, because every ball of any positive radius around 1 contains points that are not equal to 1.

3. (15 points) Prove or give a counterexample.

(a) If $\sum a_n$ converges then $\sum |a_n|$ converges.

(b) If $\sum |a_n|$ converges then $\sum a_n$ converges.

The first statement is false, as seen for example by the series $\sum \frac{(-1)^n}{\sqrt{n}}$ which converges by the theorem on alternating series, but does not converge absolutely.

The second statement is true, and can be proved using the Cauchy criterion. Let $\epsilon > 0$. Since $\sum |a_n|$ converges, there exists N such that $n \geq m > N$ implies that

$$|\sum_{k=m}^{n} |a_k|| = \sum_{k=m}^{n} |a_k| < \epsilon.$$

Using the triangle inequality, we also have that $n \geq m > N$ implies

$$|\sum_{k=m}^{n} a_k| \leq \sum_{k=m}^{n} |a_k| < \epsilon$$

so $\sum a_n$ converges.

4. (15 points) (a) Give an example of an open cover of $(0, 1) \subset \mathbb{R}$ that has no finite subcover.

(b) Let $E = \{0\} \bigcup \{\frac{1}{n} : n \in \mathbb{N}\}$. Prove that E is a compact subset of $\mathbb{R}$.

For (a), consider $\mathcal{U} = \{(\frac{1}{n+2}, \frac{1}{n}) : n \in \mathbb{N}\}$. No finite subcover of $\mathcal{U}$ can cover $(0, 1)$. To see this, fix a finite subset $\mathcal{V}$ of $\mathcal{U}$ and let m be the largest integer such that $(\frac{1}{m+2}, \frac{1}{m})$ is in $\mathcal{V}$. Then, $\mathcal{V}$ does not cover any of points $< \frac{1}{m+2}$ in $(0, 1)$.

For (b), let $\mathcal{U}$ be any open cover of E. Then, there exists a set $U_0 \in \mathcal{U}$ containing 0. Since U_0 is an open set, there exists some radius $r > 0$ such that $B_r(0) = \{x \in \mathbb{R} : |x| < r\} \subset U_0$. Since $r > 0$ and $\lim \frac{1}{n} = 0$, there exists an m such that $r > \frac{1}{m}$. Then, U_0 contains all of the infinitely many points $\{\frac{1}{n} : n \geq m\}$ of E, because $B_r(0)$ contains them.

Now there are only finitely many points $\{\frac{1}{n} : 1 \leq n < m\}$ remaining to cover. Since $\mathcal{U}$ is an open cover of E, we can select an open set $U_n \in \mathcal{U}$ containing $\frac{1}{n}$ for each $1 \leq n < m$. Together with U_0, these sets form a finite subcover of E.

5. (15 points) Let s_n be defined recursively by

$$s_1 = 7$$

and

$$s_{n+1} = \sqrt{2 + s_n}.$$

(a) Show that s_n converges. (Hint: Recall that $(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$.)

(b) Find the limit of s_n.

We begin by noting that $s_n \geq 2$ for all n by induction where $s_1 = 7 > 2$ forms a base case. Next, suppose $s_n \geq 2$. Then, $2 + s_n \geq 4$ so $s_{n+1} = \sqrt{2 + s_n} \geq 2$, proving the inductive claim. Hence, $s_n \geq 2$ for all n by the principle of mathematical induction.

Next, we claim that s_n is monotonically nonincreasing. To see this consider

$$\begin{aligned} 0 \leq (s_n - 2)(s_n + 1) = s_n^2 - s_n - 2 &= (s_n - \sqrt{2 + s_n})(s_n + \sqrt{2 + s_n}) \\ &= (s_n - s_{n+1})(s_n + s_{n+1}) \end{aligned}$$

where we are using that $s_n \geq 2$ for all n and the standard algebraic trick for clearing square roots. Since $(s_n + s_{n+1})$ is nonnegative and the product $(s_n + s_{n+1})(s_n - s_{n+1})$ is nonnegative, we must have that $(s_n - s_{n+1})$ is nonnegative. This proves that $s_{n+1} \leq s_n$, so the sequence is nonincreasing.

This implies that $s_n \leq 7$ for all n, so s_n is a bounded sequence. By the theorem that bounded monotonic sequences converge, we have that s_n converges.

To find the limit, we apply the limit theorems on both sides of the recurence:

$$\begin{aligned} \lim s_{n+1}^2 &= \lim(2 + s_n) \\ (\lim s_{n+1})(\lim s_{n+1}) &= \lim 2 + \lim s_n \\ s^2 &= 2 + s \end{aligned}$$

where $s = \lim s_n = \lim s_{n+1}$, and solving

$$0 = s^2 - s - 2 = (s - 2)(s + 1)$$

for s gives $s = 2$. (The solution $s = -1$ cannot be the limit since all of the terms s_n are positive.)

6. (15 points) Investigate the convergence or divergence of the series $\sum a_n$.

(a) $a_n = (\sqrt[n]{n} - 1)^n$.

(b) $a_n = \sqrt{n+1} - \sqrt{n}$.

For (a), we have $\lim |a_n|^{1/n} = \lim(\sqrt[n]{n} - 1) = 0$ so $\limsup |a_n|^{1/n} = 0 < 1$ and so $\sum a_n$ converges absolutely by the root test.

For (b), we have

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \geq \frac{1}{2} \frac{1}{\sqrt{n+1}}$$

and so $\sum a_n$ diverges to ∞ since $\sum \frac{1}{\sqrt{n+1}}$ diverges using the integral test.

7. (10 points) Please circle TRUE or FALSE.

a. (TRUE or FALSE) If S is a bounded subset of $\mathbb{R}^k$ then the closure $\overline{S}$ is compact.

This is true by the Heine-Borel theorem.

b. (TRUE or FALSE) The union of an infinite collection of compact sets is compact.

This is false, since points are compact and an an unbounded collection of these in $\mathbb{R}$ is not.

c. (TRUE or FALSE) The intersection of an infinite collection of compact sets in $\mathbb{R}^k$ is compact.

This is true by Heine-Borel because any intersection of closed bounded sets remains closed and bounded.

d. (TRUE or FALSE) $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$ is a metric for $\mathbb{R}$.

This is true. The only hard axiom to check is the triangle inequality

$$d(x, y) \leq d(x, z) + d(y, z)$$

but the left size is 1 only if $x \neq y$, in which case at least one of $d(x, z)$ or $d(y, z)$ is 1.

e. (TRUE or FALSE) Every sequence of points from

$$\{x \in \mathbb{R}^2 : |x_1| \leq 2 \text{ and } |x_2| \leq 1\}$$

has a convergent subsequence.

This is true because the set is compact by the Heine-Borel theorem, being both closed and bounded.