Midterm Exam

This is the midterm exam for Math 25, Fall 2007. The exam has 100 points, and you have 50 minutes to complete this exam. You may not use any notes or books, nor any calculating or computing devices. Please give *as much justification as you can* for all of your solutions.

1. (25 points) Let S be a nonempty bounded subset of $\mathbb{R}$. Give the precise definition of the greatest lower bound for S. What can you say about S if $\inf S = \sup S$?

We say that $m = \inf S$ is the greatest lower bound of S if $m \leq s$ for all $s \in S$ and whenever $m' > m$ there exists $s' \in S$ such that $s' < m'$.

Suppose that S is a set where $\inf S = m = \sup S$. Since m is a lower bound of S we have $m \leq s$ for all $s \in S$. Since m is an upper bound of S we have $m \geq s$ for all $s \in S$. Putting these together, we have

$$m \leq s \leq m$$

for all s in S, so S contains at most one element, namely m.

2. (25 points) Suppose s_n is a sequence which converges to a negative number s. Carefully prove that the terms of s_n must eventually become and stay negative.

Suppose $\lim s_n = s$ where $s < 0$. Then, choosing $\epsilon = \frac{-s}{2} > 0$ we have by the limit definition that there exists N such that $n > N$ implies

$$|s_n - s| < \frac{-s}{2}$$

so

$$s_n - s < \frac{-s}{2}$$

so

$$s_n < s + \frac{-s}{2}$$

so $s_n < \frac{s}{2} < 0$. Hence, all of the terms s_n for $n > N$ are negative.

3. (15 points) Give examples of sequences s_n and t_n such that $\lim s_n = \infty$, $\lim t_n = \infty$, and:

(a) $\lim \frac{s_n}{t_n} = 0$.
Consider $s_n = n$ and $t_n = n^2$. Then, $\lim \frac{n}{n^2} = \lim \frac{1}{n} = 0$.

(b) $\lim \frac{s_n}{t_n} = 2$.
Consider $s_n = 2n$ and $t_n = n$. Then, $\lim \frac{2n}{n} = \lim 2 = 2$.

(c) $\lim \frac{s_n}{t_n} = \infty$.
Consider $s_n = n^2$ and $t_n = n$. Then, $\lim \frac{n^2}{n} = \lim n = \infty$.

4. (25 points) Let $s_n = (-2)^{(-2)^n}$ for $n \in \mathbb{N}$. Find the $\limsup s_n$ and $\liminf s_n$ and justify your solution. Hint: Define a subsequence that converges to $\liminf s_n$.

The first few terms of the sequence are $(\frac{1}{4}, 16, \frac{1}{2^8}, 2^{16}, \dots)$. In particular, all of the terms are positive because $s_n = (-1)^{(-2)^n}(2)^{(-2)^n} = 2^{(-2)^n}$. Therefore, we have $\liminf s_n \geq 0$ because no sequence of positive terms can converge to a negative limit, for example by Problem 2 above. In fact, we claim that $\liminf s_n = 0$ because $\liminf s_n$ is the infimum of the subsequential limits of s_n, and we have that the subseqence s_{n_k} defined by $n_k = 2k - 1$ is

$$\frac{1}{2^{2^{2k-1}}}$$

which converges to 0.

On the other hand, the terms $s_{n_k} = s_{2k}$ define a subsequence $2^{2^{2k}}$ which is diverging to ∞. Since $\liminf s_n$ is the supremum of the subsequential limits of s_n, we have $\limsup s_n$ is ∞.

5. (10 points) Please circle TRUE or FALSE.

a. (TRUE or FALSE) Every sequence has an nondecreasing subsequence.

This is false; consider $s_n = -n$.

b. (TRUE or FALSE) Every sequence has a bounded subsequence.

This is false; consider $s_n = n$.

c. (TRUE or FALSE) Every bounded sequence has an monotonic subsequence.

This is true. We used this result to prove the Bolzano-Weierstrass theorem.

d. (TRUE or FALSE) Every subsequence of a bounded monotonic sequence converges.

This is true, because every bounded monontonic sequence converges.

e. (TRUE or FALSE) Every convergent sequence is bounded.

This is true. We can trap infinitely many of the sequence values in an ϵ interval about the limit point and then take the maximum of the finitely many sequence values before this point to get a bound.