

Homework 1

1.4. (a) When we evaluate the sum $f(n) = 1 + 3 + \cdots + (2n - 1)$ for low values of n we obtain $f(1) = 1$, $f(2) = 4$, $f(3) = 9$ and $f(4) = 16$. From this data, it seems reasonable to guess that $f(n) = n^2$.

(b) Let P_n be the statement “ $1 + 3 + \cdots + (2n - 1) = n^2$.” We showed that P_1, P_2, P_3 and P_4 are true in (a). For the induction step, suppose P_n is true so

$$1 + 3 + \cdots + (2n - 1) = n^2$$

and add $2n + 1 = 2(n + 1) - 1$ to both sides. We obtain

$$1 + 3 + \cdots + (2n + 1) = n^2 + (2n + 1)$$

which is

$$1 + 3 + \cdots + (2(n + 1) - 1) = (n + 1)^2$$

so P_{n+1} is true. We have shown that P_n implies P_{n+1} so the induction step holds, and the principle of mathematical induction implies that P_n is true for all n .

1.6. Let P_n be the statement “ $(11)^n - 4^n$ is divisible by 7.” The basis for induction P_1 is true since $11 - 4 = 7$. For the induction step, suppose P_n is true. To verify P_{n+1} we write

$$\begin{aligned} (11)^{n+1} - 4^{n+1} &= (11)^{n+1} - (11)4^n + (11)4^n - 4^{n+1} \\ &= ((11)^n - 4^n)(11) + 4^n(11 - 4). \end{aligned}$$

Since $(11)^n - 4^n$ is divisible by 7, and $4^n(11 - 4) = 7(4)^n$ is divisible by 7, we have that $(11)^{n+1} - 4^{n+1}$ is divisible by 7. Therefore, P_n implies P_{n+1} so the induction step holds, and the principle of mathematical induction implies that P_n is true for all n .

Polya’s Paradox. The induction step does not hold for $n = 1$. If there are $n + 1 = 2$ horses, then the induction step asks us to partition $\{1, 2\}$ into the sets $\{1\}$ and $\{2\}$ which *do not* overlap. The first base case for which the induction step works is $n = 2$, but then the statement of the theorem is obviously false: Two horses can have two different colors.