

HOMEWORK 2

2.4. Suppose for the sake of contradiction that $x = (5 - \sqrt{3})^{1/3}$ represents a rational number. Then we can write x as $\frac{p}{q}$ where p and q are integers with no common factors. Also, we have that x satisfies

$$\begin{aligned}(x^3 - 5)^2 &= 3 \\ x^6 - 10x^3 + 22 &= 0\end{aligned}$$

By the rational zeros theorem, this implies that $x \in \{\pm 1, \pm 2, \pm 11, \pm 22\}$. Hence,

$$|x^3 - 5| \geq 3$$

for all of the rational candidates for x . Thus,

$$(x^3 - 5)^2 \geq 9 > 3$$

which contradicts our first equation. Hence x is irrational.

3.4. To prove that $0 < 1$ from the axioms, consider

$$\begin{aligned}0 &\leq 1^2 && \text{by (Theorem 3.2, iv)} \\ 0 &\leq 1 && \text{by (M3) .}\end{aligned}$$

If $1 = 0$ then,

$$\begin{aligned}a &= a \cdot 1 && \text{for all } a \text{ by (M3)} \\ &= a \cdot 0 && \text{since } 1 = 0 \\ &= 0 && \text{by (Theorem 3.1, ii).}\end{aligned}$$

Since this is true for all a , there is only one element in our field, but this contradicts the definition of a field. Hence, $0 < 1$.

To prove that $0 < a < b$ implies $0 < b^{-1} < a^{-1}$ from the axioms, observe that $a > 0$ and $b > 0$ so $a^{-1} > 0$ and $b^{-1} > 0$ by Theorem 3.2, (vi). By Theorem 3.2 (iii), we have $a^{-1}b^{-1} \geq 0$. Then, $a < b$ implies

$$\begin{aligned}a(a^{-1}b^{-1}) &< b(a^{-1}b^{-1}) && \text{by (O5)} \\ (aa^{-1})b^{-1} &< (bb^{-1})a^{-1} && \text{by (M1) and (M2)} \\ 1 \cdot b^{-1} &< 1 \cdot a^{-1} && \text{by (M4)} \\ b^{-1} &< a^{-1} && \text{by (M3).}\end{aligned}$$

3.6. (a) By the triangle inequality and (A1), we have

$$|a + b + c| = |a + (b + c)| \leq |a| + |b + c|$$

and by the triangle inequality again, we have $|b + c| \leq |b| + |c|$. Hence,

$$|a + b + c| \leq |a| + |b + c| \leq |a| + |b| + |c|.$$

(b) Let P_n be the statement “ $|a_1 + \cdots + a_{n+1}| \leq |a_1| + \cdots + |a_{n+1}|$.” The triangle inequality gives $|a_1 + a_2| \leq |a_1| + |a_2|$, so P_1 is true. For the induction step, suppose P_n is true. Then, by the triangle inequality and (A1) we have:

$$|(a_1 + \cdots + a_{n+1}) + a_{n+2}| \leq |a_1 + \cdots + a_{n+1}| + |a_{n+2}|$$

which is

$$\leq |a_1| + \cdots + |a_{n+1}| + |a_{n+2}|$$

because P_n is true. Hence, we have shown that P_{n+1} is true, so the induction holds. By the principle of mathematical induction, the statement is true for all n .

3.8. We have that a, b are real numbers and $a \leq b_1$ for every $b_1 > b$. Suppose for the sake of contradiction that $a > b$ so $(a - b) > 0$. Then we may let $b_1 = b + \frac{1}{2}(a - b) = \frac{1}{2}(a + b)$ and we have $b_1 > b$. But then we have:

$$a > b = 2b_1 - a$$

$$2a > 2b_1$$

so $a > b_1$ which contradicts our first hypothesis. Hence, $a \leq b$.

4.4. The infimum of the set in part (g) is 0. The infimum of the set in part (j) is $\frac{2}{3}$. The infimum of the set in part (q) is 0. The infimum of the set in part (s) is 0. The infimum of the set in part (t) is not defined because the set is not bounded below. The infimum of the set in part (u) is 0. The infimum of the set in part (v) is -1.

4.8. (a) Since S and T are nonempty subsets of \mathbb{R} and $s \leq t$ for all $s \in S, t \in T$, we have S is bounded above by any element of T and T is bounded below by any element of S .

(b) Let $s_0 = \sup S$ and $t_0 = \inf T$. Then, choose any $t_1 > t_0$. Since t_0 is the infimum of T , there exists an element t' of T that is less than t_1 . Then, by (a) we have t' is an upper bound for S , so t_1 is also an upper bound for S because $t_1 > t'$. But then $s_0 \leq t_1$ because s_0 is the *least* upper bound of S . Hence, we have shown $s_0 \leq t_1$ for every $t_1 > t_0$, so by Exercise 3.8 we have $s_0 \leq t_0$.

(c) Consider $S = [0, 1]$ and $T = [1, 2]$. Then, $S \cap T = \{1\}$.

(d) Consider $S = [0, 1)$ and $T = (1, 2]$. Then, $S \cap T = \emptyset$, and $\sup S = 1 = \inf T$.

4.10. Suppose $a > 0$. Applying the Archimedean property twice, we have that there exist n_1 and n_2 in \mathbb{N} such that $an_1 > 1$ and $n_2 > a$. Hence, we have

$$\frac{1}{n_1} < a < n_2$$

and setting $n = \max(n_1, n_2)$, we obtain

$$\frac{1}{n} \leq \frac{1}{n_1} < a < n_2 \leq n.$$

4.14. (a) We have $S = \{a + b : a \in A, b \in B\}$ where A and B are nonempty bounded subsets of \mathbb{R} . Let $a_0 = \sup A$ and $b_0 = \sup B$. By the completeness axiom, these are well-defined real numbers. Then, $a_0 \geq a$ for all $a \in A$ and $b_0 \geq b$ for all $b \in B$, so $a_0 + b_0 \geq a + b$ for all $a + b \in S$. This gives that $a_0 + b_0$ is an upper bound for S so $\sup S \leq a_0 + b_0$.

Suppose $\sup S < a_0 + b_0$ for the sake of contradiction. Then the average

$$s_1 = \frac{1}{2}(\sup S + (a_0 + b_0))$$

satisfies

$$s_1 < a_0 + b_0.$$

Hence, $s_1 - b_0 < a_0$ which implies there exists $a' \in A$ such that $a' > s_1 - b_0$ (for otherwise $s_1 - b_0$ would be an upper bound for A that is smaller than the *least* upper bound a_0). Similarly, $s_1 - a_0 < b_0$ which implies there exists $b' \in B$ such that $b' > s_1 - a_0$. Putting these together, we have

$$a' + b' > 2s_1 - (a_0 + b_0)$$

but by our choice of s_1 this is

$$a' + b' > 2s_1 - (a_0 + b_0) = \sup S.$$

This contradicts that $\sup S$ is an upper bound for S because $a' + b' \in S$. Hence, we must have $\sup S = a_0 + b_0$.

(b) The proof is entirely similar to (a), but you should write it out for practice.