Homework 3

5.6. If S and T are bounded then the result holds by Exercise 4.7(a) (proved in class). If S is not bounded above, then T is not bounded above because $S \subset T$. Hence, if $\sup S = \infty$ then $\sup T = \infty$ and so $\sup S \leq \sup T$ in all cases. Similarly, if S is not bounded below, then T is not bounded below. Hence, if $\inf S = -\infty$ then $\inf T = -\infty$ and so $\inf T \leq \inf S$ in all cases. Finally, we have $\inf S \leq \sup S$ in all cases because we order $\mathbb{R} \cup \{-\infty, \infty\}$ by $-\infty < x < \infty$ for all $x \in \mathbb{R}$.

7.4. (a) Consider the sequence $x_n = \frac{1}{n}\sqrt{2}$. If any number in this sequence were rational, then we could write

$$\frac{1}{n}\sqrt{2} = \frac{p}{q}$$

$$\sqrt{2} = \frac{np}{q}$$

where np and q are integers. Hence, $\sqrt{2}$ would be rational which is a contradiction. This is a sequence of irrational numbers. However, the terms of the sequence become smaller and smaller, so the limit of the sequence is 0, which *is* a rational number.

(b) Consider the sequence r_n consisting of decimal approximations to n digits of $\sqrt{2}$. Any finite decimal approximation is rational; for example $r_8 = 1.41412136 = \frac{141412136}{100000000}$. On the other hand, the terms of this sequence tend to $\sqrt{2}$ by definition, which is *not* rational.

8.2. (a) $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$. Let $\epsilon > 0$ and choose $N = \frac{1}{\epsilon}$. Then, $n > N$ implies $n > \frac{1}{\epsilon}$, and $\frac{1}{n} > 0$ for all $n \in \mathbb{N}$, so

$$n + \frac{1}{n} > n > \frac{1}{\epsilon}.$$

This is

$$\frac{n^2+1}{n} > \frac{1}{\epsilon}$$

so

$$|\frac{n}{n^2+1}| = \frac{n}{n^2+1} < \epsilon.$$

(c) $\lim_{n\to\infty} \frac{4n+3}{7n-5} = \frac{4}{7}$. Let $\epsilon > 0$ and choose $N = \frac{1}{49}(\frac{41}{\epsilon} + 35)$. Then, $n > N$ implies $n > \frac{1}{49}(\frac{41}{\epsilon} + 35)$ so

$$(49n - 35)\epsilon > 41$$

and

$$\epsilon > \frac{41}{(49n-35)} = \frac{(28n+21) - (28n-20)}{7(7n-5)} = \frac{7(4n+3) - 4(7n-5)}{7(7n-5)}$$

so

$$|\frac{4n+3}{7n-5} - \frac{4}{7}| = \frac{7(4n+3) - 4(7n-5)}{7(7n-5)} < \epsilon.$$

(e) $\lim_{n\to\infty} \frac{1}{n} \sin n = 0$. Let $\epsilon > 0$ and choose $N = \frac{1}{\epsilon}$. Then, $n > N$ implies $n > \frac{1}{\epsilon}$ and so we have $\frac{1}{n} < \epsilon$. Hence,

$$|\frac{1}{n} \sin n| = \frac{1}{n}|\sin n| \leq \frac{1}{n} < \epsilon$$

because $|\sin n| \leq 1$ for all $n \in \mathbb{N}$.