HOMEWORK 4

8.4. Suppose that there exists an $M > 0$ such that $|t_n| \leq M$ for all $n \in \mathbb{N}$. To prove that $\lim(s_n t_n) = 0$ let $\epsilon > 0$ and then feed $\frac{\epsilon}{M}$ into the definition of $\lim s_n = 0$ to obtain N such that $n > N$ implies

$$|s_n| < \frac{\epsilon}{M}.$$

Then, $n > N$ implies

$$|t_n s_n| = |t_n||s_n| < M\frac{\epsilon}{M} = \epsilon.$$

8.10. Let $s = \lim_{n\to\infty} s_n$. Then, $s - a > 0$. Taking $\epsilon = \frac{1}{2}(s-a)$, we have that there exists an N such that $n > N$ implies

$$|s_n - s| < \frac{1}{2}(s-a).$$

Since this implies

$$-\frac{1}{2}(s-a) < (s_n - s) < \frac{1}{2}(s-a)$$

we have

$$s_n > s - \frac{1}{2}(s-a).$$

Then,

$$s_n > s - \frac{1}{2}(s-a) = \frac{1}{2}(s+a) = a + \frac{1}{2}(s-a) > a$$

since $s - a > 0$.

9.2. (a) Since the sequences x_n and y_n converge, we have by the addition theorem for limits that

$$\lim(x_n + y_n) = \lim(x_n) + \lim(y_n) = 3 + 7 = 10.$$

(b) Since the two sequences converge, and all of the y_n are non-zero and $\lim y_n \neq 0$, we have by the division theorem for limits that

$$\lim(\frac{3y_n - x_n}{y_n^2}) = \frac{\lim(3y_n - x_n)}{\lim(y_n^2)}$$

and now we can use the multiplication theorem for limits to reduce the denominator

$$= \frac{\lim(3y_n - x_n)}{\lim(y_n)\lim(y_n)}$$

and we can use the addition and multiplication theorems for limits to reduce the numerator

$$= \frac{3\lim(y_n) - \lim(x_n)}{\lim(y_n)\lim(y_n)} = \frac{3(7) - 3}{7^2} = \frac{18}{49}$$

9.4. (a) We have $s_n = (1, \sqrt{2}, \sqrt{\sqrt{2}+1}, \sqrt{\sqrt{\sqrt{2}+1}+1}, \ldots)$. Some decimal approximations for these terms are $(1, 1.4142, 1.5538, 1.5981)$, so it seems plausible that this sequence converges.

(b) Define a new sequence t_n by $t_n = s_{n+1}$. If we assume that s_n converges, then t_n converges also and $lim_{n\to\infty} t_n = lim_{n\to\infty} s_n$. (To see this, just let $\epsilon > 0$ and use the limit definition to obtain N_1 such that $n > N_1$ implies $|s_n - s| < \epsilon$. Setting $N = N_1 - 1$, we have $n > N$ implies $n + 1 > N_1$ so $|t_n - s| = |s_{n+1} - s| < \epsilon$.)

Now, since we *assume* that s_n converges, our limit theorems from this section apply when we take limits on both sides of

$$t_n = \sqrt{s_n + 1}$$

obtaining

$$\lim(t_n) = \lim(\sqrt{s_n + 1}).$$

Since $s_n + 1$ is nonnegative, we have by Example 8.5 (also proved in lecture) that

$$\lim(t_n) = \sqrt{\lim(s_n + 1)}$$

and by the addition theorem for limits, this gives

$$\lim(t_n) = \sqrt{\lim(s_n) + 1}$$

But since $\lim s_n = \lim t_n$, this is a quadratic polynomial that we can solve to find $s = \lim s_n$. We have

$$s = \sqrt{s + 1}$$

which is

$$s^2 - s - 1 = 0$$

so applying the quadratic formula, we find

$$s = \frac{1 \pm \sqrt{(-1)^2 - 4(-1)}}{2}.$$

Since the terms of s_n are nonnegative, the limit must be the positive root.

(Note that we have *not* proved that the sequence converges: We have only shown that *if* the sequence converges, then this value must be the limit. It turns out that this sequence does converge.)