HOMEWORK 5

9.14. Let $s_n = \frac{a^n}{n^p}$, so $\lim |\frac{s_{n+1}}{s_n}|$ is

$$\lim |a\frac{n^p}{(n+1)^p}| = \lim |a||(\frac{n}{n+1})^p| = |a| \lim |(\frac{n}{n+1})^p|$$

$$= |a| \lim((1 - \frac{1}{n+1})^p) = |a|(\lim(1 - \frac{1}{n+1}))^p$$

$$= |a|(\lim(1) - \lim(\frac{1}{n+1}))^p = |a|$$

where we used the scalar multiplication theorem for limits once, we used the multiplication theorem for limits p times, and we used the addition theorem for limits once. If $|a| < 1$ then by 9.12(a) we have that $\lim s_n = 0$. If $|a| > 1$, then $a > 1$ or $a < -1$. If $a > 1$ then all of the terms of s_n are positive, and so we have by 9.12(b) that $+\infty = \lim |s_n| = \lim s_n$. If $a < -1$ then $|s_n|$ diverges to infinity by 9.12(b), but the even terms of s_n are positive while the odd terms of s_n are negative. Therefore, the limit does not exist in this case.

9.18. This problem introduces the *geometric series*.

(a) Observe that

$$(1-a)(1+a+a^2+\cdots+a^n) = (1+a+a^2+\cdots+a^n)-(a+a^2+\cdots+a^{n+1})$$
$$= 1 - a^{n+1}$$

so if $a \neq 1$ then we can divide the left and right sides of this equation by $1 - a$ to obtain

$$1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}$$

as desired.

(b) Since $|a| \neq 1$ we can apply (a) to obtain

$$\lim(1 + a + a^2 + \cdots + a^n) = \lim(\frac{1 - a^{n+1}}{1 - a})$$

$$= \frac{1}{1-a} \lim(1 - a^{n+1}) = \frac{1}{1-a}(\lim(1) - \lim(a^{n+1}))$$

using the scalar multiplication and addition theorems for limits. Since $|a| < 1$ we have that $\lim(a^{n+1}) = 0$ by result 9.7(b) in the text, so the limit is $\frac{1}{1-a}$.

(c) By (b) this is $\frac{1}{1-\frac{1}{3}} = \frac{3}{2}$.

(d) For $a \geq 1$ we have $\lim(1 + a + a^2 + \cdots + a^n) = +\infty$ because for all $M > 0$ we can take $N = M$. Then, $n > N$ implies

$$M = \sum_{i=1}^{M} 1 < \sum_{i=1}^{M} a^i < 1 + a + a^2 + \cdots + a^n.$$

10.6. (a) To prove that s_n is Cauchy, let $\epsilon > 0$ and choose $N = 1 - \log_2 \epsilon$. Then, suppose $m, n > N$ and without loss of generality let us call the smaller index m so that $m < n < N$. We must show that $|s_n - s_m| < \epsilon$. We have $m > N = 1 - \log_2 \epsilon$ so

$$\log_2 \epsilon > 1 - m.$$

Using the notation $\sum_{i=k}^{\infty} a^i$ to mean $\lim_{n\to\infty}(a^k + a^{k+1} + a^{k+2} + \cdots + a^{k+n})$ we can use geometric series to write

$$\epsilon > 2^{1-m} = 2(2^{-m}) = \frac{1}{1-\frac{1}{2}} 2^{-m} = 2^{-m} \sum_{i=0}^{\infty} (\frac{1}{2})^i = \sum_{i=m}^{\infty} (\frac{1}{2})^i.$$

By the triangle inequality, we have

$$\begin{aligned} |s_n - s_m| &= |s_n - s_{n-1} + s_{n-1} - s_{n-2} + \cdots + s_{m+1} - s_m| \\ &\leq |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \cdots + |s_{m+1} - s_m| \end{aligned}$$

which is

$$< \sum_{i=m}^{n-1} 2^{-i}$$

by the information given in the problem. We view this series as a monotonically increasing sequence, so the limit is an upper bound and we have

$$\sum_{i=m}^{n-1} 2^{-i} < \sum_{i=m}^{\infty} 2^{-i} < \epsilon$$

where the last inequality comes from our estimate above.

(b) The proof in (a) relied on $\sum_{i=0}^{\infty} 2^{-i}$ being finite (which by definition is just the limit

$$\lim(1 + \frac{1}{2} + (\frac{1}{2})^2 + \cdots + (\frac{1}{2})^n)$$

as $n \to \infty$ that we calculate using geometric series). You may recall from Calculus that

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges to ∞, so we will not be able to prove the analogous theorem.

10.10. (a) The first few terms of the sequence are $(1, \frac{2}{3}, \frac{5}{9}, \frac{14}{27}, \dots)$.

(b) The first term satisfies $s_1 > \frac{1}{2}$. Now suppose that $s_n > \frac{1}{2}$, so $s_{n+1} = \frac{1}{3}(s_n + 1) > \frac{1}{3}(\frac{1}{2} + 1) = \frac{1}{2}$ and the induction step holds. By the principle of mathematical induction, we conclude that $s_n > \frac{1}{2}$ for all $n \in \mathbb{N}$.

(c) Observe that by (b),

$$s_n - s_{n+1} = s_n - \frac{1}{3}(s_n + 1) = \frac{2}{3}s_n - \frac{1}{3} > \frac{2}{3}(\frac{1}{2}) - \frac{1}{3} = 0,$$

so the sequence is nonincreasing. In fact, it is strictly decreasing.

(d) By the bounded monotone sequence theorem, we have that s_n converges because s_n is bounded below by (b), bounded above by $s_1 = 1$ using (c), and s_n is monotone by (c). Let $s = \lim s_n$. Then, we can apply the limit theorems and take limits on both sides of $s_{n+1} = \frac{1}{3}(s_n + 1)$ obtaining

$$\lim s_{n+1} = \lim(\frac{1}{3}(s_n + 1)) = \frac{1}{3}\lim(s_n + 1) = \frac{1}{3}(\lim(s_n) + 1).$$

Since s_{n+1} is a convergent sequence with $\lim s_{n+1} = s$, we have

$$s = \frac{1}{3}(s + 1)$$

which we can solve for s to obtain

$$s = \frac{1}{2}.$$