Homework 6

11.10. (a) Suppose s_{n_k} is a convergent subsequence of

$$s_n = (1, \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{3}, \dots).$$

Let $s = \lim_{k\to\infty} s_{n_k}$. Either $s = 0$ or $s > 0$, in which case we claim that s must have the form $\frac{1}{n}$ for $n \in \mathbb{N}$. If s is not of the form $\frac{1}{n}$ for some $n \in \mathbb{N}$ then we must have strict inequalities

$$\frac{1}{n+1} < s < \frac{1}{n}$$

for some n. Our idea is to define $\delta = \inf\{|s - \frac{1}{n}| : n \in \mathbb{N}\}$ to get a lower bound on the distances from s to potential terms of the subsequence s_{n_k}. Since s is closest to $\frac{1}{n}$ and $\frac{1}{n+1}$ among all terms of the form $\frac{1}{n}$, we have

$$\delta = \inf\{|s - \frac{1}{n}| : n \in \mathbb{N}\} = \min\{|s - \frac{1}{n}|, |s - \frac{1}{n+1}|\} > 0.$$

But then $|s_{n_k} - s| \geq \delta$ for all k because each term of s_{n_k} has the form $\frac{1}{n}$ for $n \in \mathbb{N}$, and this contradicts that $\lim_{k\to\infty} s_{n_k} = s$.

Hence, $S = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$.

(b) By theorem, we have $\limsup s_n = \sup S = 1$ and $\liminf s_n = \inf S = 0$ by (a).

12.10. There are two statements to prove. If s_n is bounded by $|s_n| \leq M$ for all $n \in \mathbb{N}$ where M is a real number, then we have

$$\limsup |s_n| \leq \sup\{|s_n| : n \in \mathbb{N}\} \leq M < \infty.$$

On the other hand, if $\limsup |s_n| = P < \infty$ then choosing $\epsilon = 1$ we obtain an N such that

$$|\sup\{|s_n| : n > N\} - P| < 1$$
$$\sup\{|s_n| : n > N\} - P < 1$$
$$\sup\{|s_n| : n > N\} < P + 1.$$

Hence, we can bound $\{|s_n| : n \in \mathbb{N}\}$ by $M = \max\{|s_1|, |s_2|, \dots, |s_N|, P+1\}$.

12.14. (a) By Theorem 12.2 we have

$$\limsup |n!|^{1/n} \geq \liminf |n!|^{1/n} \geq \liminf |\frac{(n+1)!}{n!}| = \liminf |n+1| = \infty$$

so by the lim inf/lim sup theorem, we have that $(n!)^{1/n}$ diverges to ∞.

(b) Let $s_n = \frac{1}{n^n}(n!)$. Then, $|\frac{s_{n+1}}{s_n}| = \frac{n^n}{(n+1)^n}$. for all n. We claim that $\lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \frac{1}{e}$ by the quotient theorem for limits because

$$\lim \frac{n^n}{(n+1)^n} = \lim \frac{1}{\frac{(n+1)^n}{n^n}} = \lim \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e}.$$

By Corollary 12.3, we then have

$$\lim \frac{1}{n}(n!)^{1/n} = \lim |\frac{1}{n^n}(n!)|^{1/n} = \lim |\frac{s_{n+1}}{s_n}| = e^{-1}.$$