Homework 7

13.8. (a) In $\mathbb{R}$ we have that (a, b) is an open set by definition. To see this, let x be any point in (a, b) and choose $r < \min\{x - a, b - x, 1\}$. Then, we have $\{s \in \mathbb{R} : |s - x| < r\} \subset (a, b)$. Note that if a is $-\infty$ or b is $+\infty$ then this argument still works.

The set $[a, b]$ is $\mathbb{R} - \{(-\infty, a) \cup (b, \infty)\}$ so it is the complement of the union of two open sets. The union of two open sets is open, so the complement is closed.

To prove that the interior of $[a, b]$ is (a, b) we first observe that a and b are not interior points. Specifically, there is no $r > 0$ such that $B_r(a) = \{s \in \mathbb{R} : |s - a| < r\} \subset [a, b]$ because $B_r(a)$ always includes points of the form $a - \epsilon < a$ for $0 < \epsilon < r$. A similar argument shows that b is not an interior point. On the other hand, (a, b) is open so all of the points in (a, b) are interior points of (a, b). This means that for every $x \in (a, b)$ there exists $r > 0$ such that $B_r(x) \subset (a, b) \subset [a, b]$, so x is an interior point of $[a, b]$. Hence, (a, b) is the interior of $[a, b]$.

If $E = (a, b)$ then $\overline{E} = [a, b]$ and $E^\circ = (a, b)$ so the boundary of E is $\overline{E} - E^\circ = \{a, b\}$. If $F = [a, b]$ then $\overline{F} = [a, b]$ and $F^\circ = (a, b)$ so the boundary of F is $\overline{F} - F^\circ = \{a, b\}$.

(b) Let $B_r(x_0) = \{x \in \mathbb{R}^k : d(x, x_0) < r\}$ be an "open ball" in $\mathbb{R}^k$. We claim this is an open set. For any $x \in B_r(x_0)$, let $r' = \min\{r - d(x, x_0), d(x, x_0)\}$. We have $r' > 0$ since $d(x, x_0) < r$ by definition. Moreover, $B_{r'}(x) \subset B_r(x_0)$ because for any point $y \in B_{r'}(x)$ we have

$$d(y, x_0) \le d(y, x) + d(x, x_0) < r' + d(x, x_0) \le r - d(x, x_0) + d(x, x_0) = r$$

by the triangle inequality.

On the other hand, the "closed ball" given by $\overline{B_r(x_0)} = \{x \in \mathbb{R}^k : d(x, x_0) \le r\}$ is a closed set. To see this, consider the complement $C = \mathbb{R}^k - \overline{B_r(x_0)} = \{x \in \mathbb{R}^k : d(x, x_0) > r\}$. If $x \in C$ then choosing $r' = d(x, x_0) - r$ gives $B_{r'}(x) \subset C$ because for any point $y \in B_{r'}(x)$ we have

$$d(y, x_0) \ge d(x, x_0) - d(y, x) > d(x, x_0) - (d(x, x_0) - r) = r$$

by the triangle inequality.

We have that $\overline{B_r(x_0)}$ is closed and any smaller set containing $B_r(x_0)$ is not closed. This follows from the proposition on closed sets because any point x of $\{x : d(x, x_0) = r\}$ has a sequence of points in $B_r(x_0)$ that converge to x, for example $x^{(n)} = x + \frac{1}{n}(x_0 - x)$. Hence, the closure of $B_r(x_0)$ is $\overline{B_r(x_0)}$ and so the boundary of $B_r(x_0)$ is $\{x : d(x, x_0) = r\}$. A similar argument shows that this is also the boundary of $\overline{B_r(x_0)}$.

In $\mathbb{R}^2$ the set $S = \{(x_1, x_2) : x_1 > 0\}$ is open because for any $(x_1, x_2) \in S$ we can choose $r = x_1$ and then $B_r((x_1, x_2)) \subset S$. The set $T = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$ is open because for any $(x_1, x_2) \in T$ we can choose $r = \min(x_1, x_2)$ and then $B_r((x_1, x_2)) \subset T$. A similar argument shows that the sets obtained from S and T by replacing $>$ with $\geq$ are closed, by considering the complements.

The set $U = \{(x_1, x_2) : x_1 > 0, x_2 \geq 0\}$ is neither open nor closed. It is not open because there is no ball of any radius around the point $(1, 0)$ that fits completely inside U. It is not closed because the complement is another set of the same form, which is not open.

13.10. (a) Let $\frac{1}{m}$ be a point in the set $S = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$. Then, for any $r > 0$ the set $B_r(\frac{1}{m}) = \{x \in \mathbb{R} : d(x, \frac{1}{m}) < r\}$ includes points that are not of the form $\frac{1}{n}$ for any n. Specifically, we can choose $k \in \mathbb{N}$ so that $\frac{\sqrt{2}}{k} < r$ and then $\frac{1}{m} + \frac{\sqrt{2}}{k}$ is in $B_r(\frac{1}{m})$ but is not even rational, so cannot be of the form $\frac{1}{n}$ for any $n \in \mathbb{N}$.