Homework 8

14.2. (a) This diverges by comparing

$$\frac{n-1}{n^2} \geq \frac{1/2n}{n^2} = \frac{1}{2n} \text{ for all } n \geq 2$$

because $\sum \frac{1}{2n}$ diverges.

(b) This does not coverge by the corollary because $\lim(-1)^n \neq 0$. It does not diverge to ∞ or $-\infty$ because it is bounded by $|\sum(-1)^n| \leq 1$.

(c) This converges. In fact, this is

$$\sum \frac{3n}{n^3} = 3\sum \frac{1}{n^2} = \frac{\pi^2}{2}.$$

(d) This converges absolutely by the ratio test since

$$|\frac{a_{n+1}}{a_n}| = |\frac{(n+1)^3}{3^{n+1}}\frac{3^n}{n^3}| = |\frac{1}{3}(\frac{n+1}{n})^3|$$

so $\lim|\frac{a_{n+1}}{a_n}| = \frac{1}{3}$ and hence $\limsup|\frac{a_{n+1}}{a_n}| = \frac{1}{3} < 1$.

(e) This converges absolutely by the ratio test since

$$|\frac{a_{n+1}}{a_n}| = |\frac{(n+1)^2}{(n+1)!}\frac{n!}{n^2}| = |\frac{n+1}{n^2}|$$

so $\lim|\frac{a_{n+1}}{a_n}| = 0$ and hence $\limsup|\frac{a_{n+1}}{a_n}| = 0 < 1$.

(f) This converges by comparing

$$\frac{1}{n^n} \leq \frac{1}{n^2} \text{ for all } n \geq 1$$

because $\sum \frac{1}{n^2}$ converges.

(g) This converges absolutely by the root test since

$$|a_n|^{1/n} = (\frac{n}{2^n})^{1/n} = \frac{1}{2}n^{1/n}$$

so $\lim|a_n|^{1/n} = \frac{1}{2}$ and hence $\limsup|a_n|^{1/n} = \frac{1}{2} < 1$.

14.6. (a) Suppose that $|b_n| \leq M$ for all $n \in \mathbb{N}$. We show that $\sum a_n b_n$ converges by the Cauchy criterion. Let $\epsilon > 0$. Since $\sum |a_n|$ converges, it satisfies the Cauchy criterion. Therefore there exists $N \in \mathbb{N}$ such that $n \geq m > N$ implies that

$$\sum_{k=m}^{n} |a_k| < \frac{\epsilon}{M}.$$

Then, $n \geq m > N$ also implies that

$$|\sum_{k=m}^{n} a_k b_k| \leq \sum_{k=m}^{n} |a_k||b_k| \leq M\sum_{k=m}^{n} |a_k| < M\frac{\epsilon}{M} = \epsilon$$

using the triangle inequality and the boundedness of b_n. Hence, $\sum a_n b_n$ satisfies the Cauchy criterion and so converges.

(b) Let a_n be absolutely convergent and b_n be the constant sequence 1 which is bounded. Then (a) implies that $\sum a_n b_n = \sum a_n$ converges.

15.6. (a) We have seen that $\sum \frac{1}{n}$ diverges yet $\sum \frac{1}{n^2}$ converges.

(b) If $\sum a_n$ is a convergent series of nonnegative terms then $\lim a_n = 0$ and so there must exist N such that $a_n < 1$ for all $n > N$. Then $n > N$ implies that $a_n^2 < a_n < 1$ so $\sum_{N+1}^{\infty} a_n^2$ converges by the comparison test. Hence,

$$\sum_0^\infty a_n^2 = \sum_0^N a_n^2 + \sum_{N+1}^\infty a_n^2$$

converges as well since the first sum is finite.

(c) The restriction to nonnegative terms is crucial. We have by the Alternating Series Theorem that $\sum \frac{(-1)^n}{\sqrt{n}}$ converges but

$$\sum (\frac{(-1)^n}{\sqrt{n}})^2 = \sum \frac{1}{n}$$

does not.