MAT168
TAKE-HOME FINAL
Wednesday June 4, 2008

Due Wednesday June 11, 2008

Instructions :

1. You are welcome to consult your notes from class or the book, but refrain from using other sources.

2. Answer each question on a separate sheet of paper; be sure to label the paper with the number of problem and please put your answers in the same order as the questions below. If you can't answer a question, then leave your solution blank.

3. Turn your exam into my box in the math department mail room (in MSB). Please note that the mail room is only accessible from 9:00 AM to 12:10 PM and 1:00 PM to 4:00 PM Monday through Friday.

4. **Late exams will not be accepted!!!!** Turn it whatever you have finished by June 11, 2008.

5. You may use a computer to solve any linear program on this exam. You might wish to use MATLAB's linprog command, your own project, or the primitive pivot_step.m routine (or, for that matter, any other method you would like).

Good Luck!

Problem 1. Find a 4×2 matrix A, a vector b of length 2, and a vector c of length 2 for which the linear programs

$$\begin{aligned} \text{max:} \quad & c^t x \\ \text{subject to:} \quad & Ax \leq b \\ & 0 \leq x \end{aligned} \tag{1}$$

and

$$\begin{aligned} \text{max:} \quad & c^t x \\ \text{subject to:} \quad & Ax \leq b \\ & \text{x free} \end{aligned} \tag{2}$$

are both bounded and feasible AND, for which the solution of (2) is strictly greater than the solution of (1).

(a) Sketch a picture of the feasible region of (1) and indicate a vertex on the boundary at which the problem (1) has a solution.

(b) Sketch a picture of the feasible region of (2) and indicate a vertex on the boundary at which the problem (2) has a solution.

(c) Is it possible for the solution of (1) to be larger than the solution of (2)? Explain why or why not.

Note: this problem does not require computation at all if you are clever about picking A.

Problem 2. Consider a linear program of the form

$$\begin{aligned} \text{max:} \quad & c^t x \\ \text{subject to:} \quad & Ax = b \\ & 0 \leq x \leq u \end{aligned} \tag{3}$$

where A is an $m \times n$ matrix of rank m, c and b are aribitrary vectors of length n, and u is a vector of length n with nonnegative entries.

(a) Show that the dual of a problem of the form (3) is always feasible.

(b) Give an example of an 3×6 matrix A of **rank m** and a vector $b \in \mathbb{R}^m$ such that the linear program (3) is infeasible for any choice of u.

(c) Show that if the program (3) is infeasible, then its dual is unbounded.

Problem 3. Given an $m \times n$ matrix A, we say that a vector

$$x^* = \begin{pmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{pmatrix}$$

is a minimum l^1 norm solution of $Ax = b$ if $Ax^* = b$ and

$$|x_1^*| + |x_2^*| + \ldots + |x_n^*| \leq |x_1| + |x_2| + \ldots + |x_n|$$

for all solutions x of $Ax = b$.

(a) Write down a linear program for finding a minimum l_1 norm solution of a linear system of equations $Ax = b$ where A is an $m \times n$ matrix.

(b) Now let A be the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & -1 & 0 & -1 & 3 \\ 1 & 1 & -1 & 3 & -6 \end{pmatrix}$$

and let b be the vector

$$b = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}.$$

Find a minimum l^1 norm solution of the equation $Ax = b$ using the simplex method. That is, write down an optimal feasible vector for the linear program.

Problem 4. (a) Show that Newton's Method can be used to approximate

$$\frac{1}{b},$$

where b is a positive constant, without performing any division operations. (Hint: consider the function $f(x) = 1/(xb) - 1$.)

(b) Use Newton's Method to approximate $(2)^{1/3}$ (the real-valued third root of 2). Start with the initial guess of $x_0 = 6/5$ and compute three iterates x_1, x_2, and x_3.

All of the computations for this part must be done in **EXACT** arithmetic (i.e., not using approximate computer arithmetic). Write each iterate as a fraction.

Problem 5. The diagram in Figure 1 defines a minimum-cost network flow problem.

(a) Write down the linear program for this network flow problem.

(b) Find a solution (i.e., an optimal basic feasible vector) with integer entries for the program using the simplex method. A solution consists of a flow value for each arc in the network; indicate clearly the value of your solution at each arc in the newtok.

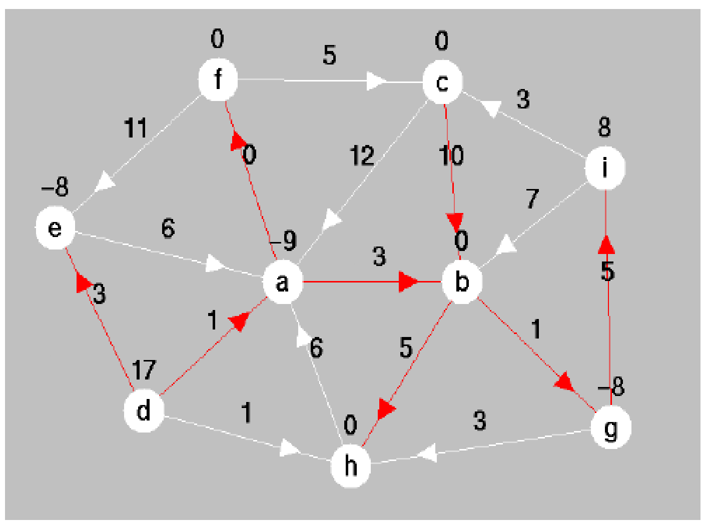

FIGURE 1. The numbers above the nodes are supplies (negative values represent demands) and the numbers shown above arcs are shipping costs.