MAT22A SECTION 2
MIDTERM ONE AND SOLUTIONS
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Problem 1. (20 pts) Consider the following system of linear equations:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 2 \\ 2x_1 + 2x_2 + 3x_3 + 5x_4 &= 3 \\ 3x_1 + 3x_2 + 4x_3 + 7x_4 &= -1 \\ x_1 + x_2 + 2x_3 + 4x_4 &= 1 \end{aligned}$$

(a) Write down the augmented matrix representing the system.

(b) Clearly state whether the system has no solutions, one solution, or many solutions.

(c) Write down all solutions of the system (if any).

Solution. (a) The augmented matrix for this system is

$$A = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 5 & 3 \\ 3 & 3 & 4 & 7 & -1 \\ 1 & 1 & 2 & 4 & 1 \end{array}\right)$$

In order to answer (b) and (c), we row reduce the augmented matrix:

$$\begin{aligned} A = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 5 & 3 \\ 3 & 3 & 4 & 7 & -1 \\ 1 & 1 & 2 & 4 & 1 \end{array}\right) &\simeq \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & 3 & -1 \end{array}\right) \\ &\simeq \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 6 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ &\simeq \left(\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ &\simeq \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -9 \\ 0 & 0 & 1 & 0 & 17 \\ 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{aligned}$$

(b) From the row reduction we just performed, we see that the system of equations has an an infinite number of solutions (4 variables and 3 constraints).

(c) We choose x_2 as our free variable and set $x_2 = t$. Then we see from that all solutions of the system are of the form

$$\begin{aligned} x_1 &= -9 - t \\ x_2 &= t \\ x_3 &= 17 \\ x_4 &= -6. \end{aligned}$$

Problem 2. (15 pts) Let

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 2 & -3 & 4 \\ 1 & 1 & -2 & 2 \\ 1 & -1 & 4 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 0 & 1 & 1 \\ -2 & 2 & -1 & -2 \\ -1 & 1 & 1 & 2 \\ -3 & 4 & -1 & 1 \end{pmatrix}.$$

(a) Compute $\det(A)$.

(b) Compute $\det(B)$.

(c) Compute $\det(AB)$.

(d) Compute $\det(A+B)$.

Solution. (a) We could use cofactor expansion to compute the determinant of A, but since it is 4×4 it is probably faster to use row and column operators. Here we use a sequence of row operations to reduce A to upper triangular form:

$$\begin{vmatrix} 1 & 0 & 0 & -1 \\ 2 & 2 & -3 & 4 \\ 1 & 1 & -2 & 2 \\ 1 & -1 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & -3 & 6 \\ 0 & 1 & -2 & 3 \\ 0 & -1 & 4 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & 2 & 5 \\ 0 & -1 & 4 & 2 \end{vmatrix}$$
$$= -\begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & 2 & 5 \\ 0 & 1 & -4 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -4 & -2 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 5 & 10 \end{vmatrix}$$
$$= 5\begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -4 & -2 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{vmatrix} = 5\begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -4 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix}$$
$$= -5\begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -4 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -5.$$

(b) We will reduce B to an upper triangular form using a sequence of row operations:

$$\begin{vmatrix} -1 & 0 & 1 & 1 \\ -2 & 2 & -1 & -2 \\ -1 & 1 & 1 & 2 \\ -3 & 4 & -1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 1 & 1 \\ 0 & 2 & -3 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 4 & -4 & -2 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 1 & 1 \\ 0 & 0 & -3 & -6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -4 & -6 \end{vmatrix}$$
$$= -3\begin{vmatrix} -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -4 & -6 \end{vmatrix} = -3\begin{vmatrix} -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 3\begin{vmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$
$$= 3(-1)(1)(1)(2) = -6$$

(c) Now that we have computed the determinant of A and B, we know the determinant of their product: $\det(AB) = \det(A)\det(B) = (-5)(-6) = 30$

(d) We first compute the sum of A and B:

$$A + B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 4 & -4 & 2 \\ 0 & 2 & -1 & 4 \\ -2 & 3 & 3 & 2 \end{pmatrix}.$$

Performing a cofactor expansion down the first row of $A + B$ gives us:

$$|A + B| = \begin{vmatrix} 0 & 4 & 2 \\ 0 & 2 & 4 \\ -2 & 3 & 2 \end{vmatrix}.$$

Now a cofactor expansion down the first column of this matrix gives us

$$\begin{aligned} |A + B| &= \begin{vmatrix} 0 & 4 & 2 \\ 0 & 2 & 4 \\ -2 & 3 & 2 \end{vmatrix} \\ &= -2 \begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix} \\ &= -2(4 * 4 - 2 * 2) = -24. \end{aligned}$$

Problem 3. (20 pts) Let A be the 3×3 matrix

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 3 & c \\ 2c & c & 1 \end{pmatrix}$$

(a) Find the values of c for which A is singular.

(b) Compute the inverse of A for all values of c for which A is nonsingular (your answer will depend on c).

Solution. We will do both parts simultaneously by row reducing the augmented matrix $\left(A \mid I \right)$. Note that it is perfectly acceptable to compute the determinant of A for part (a) and then perform a row reduction for part (b).

$$\left(\begin{array}{ccc|ccc} 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 3 & c & 0 & 1 & 0 \\ 2c & c & 1 & 0 & 0 & 1 \end{array}\right) \simeq \left(\begin{array}{ccc|ccc} 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & c-1 & -1 & 1 & 0 \\ 0 & -c & 1-c & -c & 0 & 1 \end{array}\right)$$

$$\simeq \left(\begin{array}{ccc|ccc} 2 & 0 & 3-2c & 3 & -2 & 0 \\ 0 & 1 & c-1 & -1 & 1 & 0 \\ 0 & 0 & c^2-2c+1 & -2c & c & 1 \end{array}\right)$$

$$\simeq \left(\begin{array}{ccc|ccc} 2 & 0 & 3-2c & 3 & -2 & 0 \\ 0 & 1 & c-1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2c/(c-1)^2 & c/(c-1)^2 & 1/(c-1)^2 \end{array}\right)$$

$$\simeq \left(\begin{array}{ccc|ccc} 2 & 0 & 3-2c & 3 & -2 & 0 \\ 0 & 1 & 0 & (c+1)/(c-1) & -1/(c-1) & -1/(c-1) \\ 0 & 0 & 1 & -2c/(c-1)^2 & c/(c-1)^2 & 1/(c-1)^2 \end{array}\right)$$

$$\simeq \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & (3-c^2)/(c-1)^2 & (2c^2+c-2)/(c-1)^2 & (2c-3)/(c-1)^2 \\ 0 & 1 & 0 & (c+1)/(c-1) & -1/(c-1) & -1/(c-1) \\ 0 & 0 & 1 & -2c/(c-1)^2 & c/(c-1)^2 & 1/(c-1)^2 \end{array}\right)$$

$$\simeq \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2(3-c^2)/(c-1)^2 & 1/2(c-2)/(c-1)^2 & 1/2(2c-3)/(c-1)^2 \\ 0 & 1 & 0 & (c+1)/(c-1) & -1/(c-1) & -1/(c-1) \\ 0 & 0 & 1 & -2c/(c-1)^2 & c/(c-1)^2 & 1/(c-1)^2 \end{array}\right)$$

(a) After the second row reduction we have enough information to answer part (a). We reduced the matrix A to an upper triangular matrix by adding multiples of one column to another. That means that

$$\begin{aligned} \det(A) &= \det \begin{pmatrix} 2 & 0 & 3-2c \\ 0 & 1 & c-1 \\ 0 & 0 & c^2-2c+1 \end{pmatrix} \\ &= 2(c^2-2c+1). \end{aligned}$$

It follows that A will be singular if and only if $(c-1)(c-1) = c^2-2c+1 = 0$ which occurs if and only if $c = 1$.

(b) From the row reduction above we see that for $c \neq 1$, the inverse of A exists and it is the matrix

$$\begin{pmatrix} 1/2(3-c^2)/(c-1)^2 & 1/2(c-2)/(c-1)^2 & 1/2(2c-3)/(c-1)^2 \\ (c+1)/(c-1) & -1/(c-1) & -1/(c-1) \\ -2c/(c-1)^2 & c/(c-1)^2 & 1/(c-1)^2 \end{pmatrix}.$$

Problem 4. (15 pts) Let A and B be the matrices

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

(a) Is A in row reduced echelon form?

(b) Is B in row reduced echelon form?

(c) Write down all solutions of the linear system $Ax = b$ where

$$b = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

(d) Write down all solutions of the linear system $Bx = c$ where

$$c = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

Solution. (a) No. There are nonzero entries above the leading one in the third column of A.

(b) Yes.

(c) The system $Ax = b$ is

$$\begin{aligned} x_1 + x_3 &= 1 \\ x_2 - 3x_3 &= 2 \\ x_3 &= 1. \end{aligned}$$

This is equivalent to

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 5 \\ x_3 &= 1 \end{aligned}$$

(d) The system $Bx = c$ is

$$\begin{aligned} x_1 - 2x_3 &= -2 \\ x_2 + 3x_3 &= 1 \end{aligned}$$

We choose x_3 to be the free variable. Set $x_3 = t$. Then all solutions of the system $Bx = c$ are characterized by

$$\begin{aligned} x_1 &= -2 + 2t \\ x_2 &= 1 - 3t \\ x_3 &= t. \end{aligned}$$

Problem 5. (15 pts) A square matrix M is called skew-symmetric if $M = -M^t$. Show that if A and B are $n \times n$ skew-symmtric matrices then the matrix $C = AB - BA$ is also skew-symmetric.

Solution. We want to show that $-C^{(t)} = C$. To that end, we write (keep in mind that $(AB)^t = B^t A^t$):

$$\begin{aligned} -C^t &= -(AB - BA)^t \\ &= -((AB)^t - (BA)^t) \\ &= -(B^t A^t - A^t B^t). \end{aligned}$$

Now we use the fact that $A^t = -A$ and $B^t = -B$:

$$\begin{aligned} -C^t &= -(B^t A^t - A^t B^t) \\ &= -(-B(-A) - (-A)(-B)) \\ &= -(BA - AB) \\ &= AB - BA \\ &= C \end{aligned}$$

Problem 6. (15 pts) Let A be the 3×3 matrix given by

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 1 & 2 \end{pmatrix}.$$

(a) Find the inverse of A.

(b) Using the inverse of A, find the solution to the system of equations

$$Ax = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

Solution. (a) In order to find the inverse of A, we row reduce the augmented matrix $\left(\begin{array}{c|c} A & I \end{array} \right)$:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \simeq \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$\simeq \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

$$\simeq \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -2 & 3 & -3 \\ 0 & -1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

$$\simeq \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

(b) We can now compute the solution to $Ax = b$ by multiplying both sides of the equation by A^{-1}:

$$\begin{aligned} x &= A^{-1}b \\ &= \begin{pmatrix} -2 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix} \end{aligned}$$