Mat22A: Practice Exam 1 and Solutions

Problem 1. (20 pts) Find all solutions (if any) of the system of equations

$$\begin{aligned} 3x_1 + 3x_2 + 2 + 6x3 &= 3 \\ 3x_1 + x_2 + x_3 &= 1 \\ 2x_1 + 4x_2 + 9x_3 &= 4. \end{aligned}$$

State clearly whether there are no solutions, one unique solution, or an infinite number of solutions.

Solution. Note the 2 on the left hand side of the first equation. We write down the augmented matrix A which represents this system:

$$\left(\begin{array}{ccc|c} 3 & 3 & 6 & 1 \\ 3 & 1 & 1 & 1 \\ 2 & 4 & 9 & 4 \end{array}\right).$$

Now we row reduce the augmented matrix with the sequence of row operations:

$$\begin{aligned} \left(\begin{array}{ccc|c} 3 & 3 & 6 & 1 \\ 3 & 1 & 1 & 1 \\ 2 & 4 & 9 & 4 \end{array}\right) &\simeq \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1/3 \\ 3 & 1 & 1 & 1 \\ 2 & 4 & 9 & 4 \end{array}\right) \\ &\simeq \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1/3 \\ 0 & -2 & -5 & 0 \\ 0 & 2 & 5 & 10/3 \end{array}\right) \\ &\simeq \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1/3 \\ 0 & 0 & 0 & 10/3 \\ 0 & 2 & 5 & 10/3 \end{array}\right) \\ &\simeq \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1/3 \\ 0 & 1 & 5/2 & 5/3 \\ 0 & 0 & 0 & 10/3 \end{array}\right) \\ &\simeq \left(\begin{array}{ccc|c} 1 & 0 & 5/2 & -4/3 \\ 0 & 1 & 5/2 & 5/3 \\ 0 & 0 & 0 & 10/3 \end{array}\right). \end{aligned}$$

This last matrix is in RREF. There are no solutions to the system of equations because the corresponding matrix, when reduced to RREF form, has a row of zeros. Note we could have stopped after the third step and made the same observation.

Problem 2. (20 pts) Let A be the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 4 & 1 \\ 1 & 2 & -1 \end{pmatrix}$$

(a) Compute A^{-1}.

(b) Use the inverse to solve the system of equations $Ax = b$ where

$$b = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}.$$

Solution. We first compute A^{-1} by row reducing the augmented matrix $\left(\begin{array}{c|c} A & I \end{array} \right)$:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{array} \right) \simeq \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & -3 & 1 & 0 \\ 0 & 1 & -3 & -1 & 0 & 1 \end{array} \right)$$
$$\simeq \left(\begin{array}{ccc|ccc} 1 & 0 & 7 & 4 & -1 & 0 \\ 0 & 1 & -5 & -3 & 1 & 0 \\ 0 & 0 & 2 & 2 & -1 & 1 \end{array} \right)$$
$$\simeq \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 5/2 & -7/2 \\ 0 & 1 & 0 & 2 & -3/2 & 5/2 \\ 0 & 0 & 1 & 1 & -1/2 & 1/2 \end{array} \right)$$

The inverse of A is then

$$A^{-1} = \begin{pmatrix} -3 & 5/2 & -7/2 \\ 2 & -3/2 & 5/2 \\ 1 & -1/2 & 1/2 \end{pmatrix},$$

and we can compute the solution to $Ax = b$ by computing

$$\begin{aligned} x &= A^{-1}b \\ &= \begin{pmatrix} -3 & 5/2 & -7/2 \\ 2 & -3/2 & 5/2 \\ 1 & -1/2 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -37/2 \\ 25/2 \\ 9/2 \end{pmatrix}. \end{aligned}$$

Problem 3 (20 pts). Suppose that A and B are $n \times n$ matrices such that $|A| = 3$ and $|B| = 2$.
(a) Compute $|AB^{-1}|$.
(b) Compute $|B^t A^{-2}|$.
(c) Give an example to show that there are 2×2 matrices A and B such that $|A| = 3$ and $|B| = 2$, but $|A + B| \neq 5$.

Solution
(a) $|AB^{-1}| = |A||B^{-1}| = |A| \cdot \frac{1}{|B|} = 3 \cdot 1/2 = 3/2$
(b) $|B^t A^{-2}| = |B^t| \cdot |A^{-2}| = |B| \cdot |A|^{-2} = 2 \cdot 3^{-2} = 2/9$.
(c) There are many such examples. Let

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $|A| = 3$ and $|B| = 2$ but

$$|A + B| = \begin{vmatrix} 4 & -1 \\ 1 & 2 \end{vmatrix} = 9.$$

Problem 4 (20 pts). Find all constants c such that the matrix

$$A = \begin{pmatrix} 4 & 1 & c \\ 2 & c & 3 \\ 4 & c & 2 \end{pmatrix} \tag{1}$$

is invertible.

Solution. The matrix A is invertible if and only if $\det(A)$ is nonzero, so let's compute the determinant of A. (Note: this problem can also be done via row reduction). We compute $\det(A)$ via cofactor expansion down the first column:

$$\begin{aligned}\det(A) &= 4\begin{vmatrix} c & 3 \\ c & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & c \\ c & 2 \end{vmatrix} + 4\begin{vmatrix} 1 & c \\ c & 3 \end{vmatrix} \\ &= 4(2c - 3c) - 2(2 - c^2) + 4(3 - c^2) \\ &= -2c^2 - 4c + 8.\end{aligned}$$

Now we solve for all c such that $\det(A) = 0$ using the quadratic equation. We find that $\det(A) = 0$ if and only if

$$c = -1 + \sqrt{5} \quad \text{or} \quad c = -1 - \sqrt{5}. \tag{2}$$

So the matrix A is invertible for all c not equal to $(-1 + \sqrt{5})$ or $(-1 - \sqrt{5})$.

Problem 5 (20 pts). Given the matrices $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 & 1 \\ 2 & 3 & 1 \end{pmatrix}$, compute:

(a) AB

(b) BA

(c) B^tA^t.

Solution.

(a)

$$AB = \begin{pmatrix} 4 & 2 & 2 \\ 6 & 1 & 3 \\ 0 & -4 & 0 \end{pmatrix}$$

(b)

$$BA = \begin{pmatrix} 1 & 0 \\ 9 & 4 \end{pmatrix}$$

(c)

$$B^tA^t = \begin{pmatrix} 4 & 6 & 0 \\ 2 & 1 & -4 \\ 2 & 3 & 0 \end{pmatrix}$$