NOTES ON COHERENT STATES

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Contents

1. Canonical Coherent States

We start with a brief description of the coherent states generated by a canonical annihilation and creation operators a and a^*. They satisfy canonical commutation relation $[a, a^*] = I$. We introduce the vacuum state $|0\rangle$ with the property

$$a\,|0\rangle = 0$$

and define the state space as that spanned by repeated action of a^* on $|0\rangle$.

The canonical coherent states are defined for each complex number $z \in \mathbb{C}$ by unitary transformation of the vacuum state

$$\begin{aligned} |z\rangle &= e^{za^*-\bar{z}a}\,|0\rangle \\ &= e^{-|z|/2}e^{za^*}e^{-\bar{z}a}\,|0\rangle \\ &= e^{-|z|/2}e^{za^*}\,|0\rangle \\ &= e^{-|z|/2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}z^n\,|n\rangle\,. \end{aligned} \tag{1.1}$$

Here to obtain the second line we used the Baker-Campbell-Hausdorff formula

$$e^{A+B} = a^{-1/2[A,B]}e^Ae^B,$$

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when $[A, B]$ commutes with both A and B. In the last line we introduced the orthonormal vectors $|n\rangle = \frac{1}{\sqrt{n!}}(a^*)^n |0\rangle$, which are the eigenstates of the number operator $N = a^* a$.

From the definition it follows that for any two complex numbers z_1 and z_2

$$\begin{aligned} \langle z_1| \, |z_2\rangle &= e^{-1/2|z_2|^2 - 1/2|z_1|^2} \sum_{n=0}^{\infty} \frac{1}{n!} z_1^n \bar{z_2}^n \langle n| \, |n\rangle \\ &= e^{-\frac{1}{2}|z_1|^2 + z_1 \bar{z_2} - \frac{1}{2}|z_2|^2}. \end{aligned} \tag{1.2}$$

This shows that the states $|z\rangle$ are not orthogonal.

Let us denote $d^2 z = d(\operatorname{Re} z) d(\operatorname{Im} z)$, then consider

$$\begin{aligned} &\pi^{-1} \int |z\rangle \langle z| \, d^2 z \\ &= \pi^{-1} \sum_{n,m} \frac{1}{\sqrt{n! m!}} \int e^{-|z|^2} \bar{z}^n z^m |m\rangle \langle n| \, d(\operatorname{Re} z) d(\operatorname{Im} z). \end{aligned}$$

Introduce polar coordinates $z = |z| e^{i\theta}$, then $d^2 z = |z| d|z| d\theta$. continuing the calculation

$$= \pi^{-1} \sum_{n,m} \frac{1}{\sqrt{n! m!}} \int e^{-|z|^2} |z|^{n+m} e^{i\theta m - n} |m\rangle \langle n| \, |z| d|z| d\theta.$$

Since $\int_0^{2\pi} e^{i\theta(m-n)d\theta = \delta_{n,m} 2\pi}$ and $d|z|^2 = 2|z|d|z|$ the last expression simplifies

$$\begin{aligned} &= \sum_n \frac{1}{n!} \int e^{-|z|^2} |z|^{2n} |n\rangle \langle n| \, d|z|^2 \\ &= \sum_n |n\rangle \langle n| = I. \end{aligned}$$

Here we used that $\int_0^\infty e^{-x} x^n dx = n!$.

Therefore we have **the resolution of the identity**

$$\pi^{-1} \int |z\rangle \langle z| \, d^2 z = I. \tag{1.3}$$

1.1. **Eigenproperties of $|z\rangle$.** From the CCR it is clear that $e^{-za^*} a e^{z^*} = a + z$. Then from the definition of the canonical coherent states

$$\begin{aligned} a \, |z\rangle &= e^{-1/2|z|^2} a e^{za^*} |0\rangle \\ &= e^{-\frac{1}{2}|z|^2} e^{za^*} (a + z) |0\rangle \\ &= e^{-\frac{1}{2}|z|^2} e^{za^*} |0\rangle \\ &= z \, |z\rangle \,, \end{aligned}$$

since $a \, |0\rangle = 0$.

It follows that $\langle z| \, a \, |z\rangle = z$, so we may interpret the label z by saying it is the mean of a in the coherent state $|z\rangle$.

1.2. **Diagonal representation of operators.** For any bounded operator or a polynomial in a and a^*, $\mathcal{B}$ we may calculate

$$\langle z|\,\mathcal{B}\,|z'\rangle = e^{-\frac{1}{2}|z|^2-\frac{1}{2}|z'|^2}\sum_{n,m}\frac{1}{\sqrt{n!m!}}\langle n|\,\mathcal{B}\,|m\rangle\,\bar{z}^n z'^m. \tag{1.4}$$

This series defined an entire function of two variables $\bar{z}$ and z'. It is uniquely determined by its diagonal elements $z = z'$ [2]. For example, every monomial $\bar{z}^n z^m$ can be written as a polynomial in x and y, $z = x + iy$ and then uniquely extended to $\bar{z}^n z'^m$, by determining the coefficients in $\bar{z}^n z^m$ and use them to construct $\bar{z}^n z'^m$. Note that for the conventional basis this is never true.

Define **the lower symbol** of the operator

$$b(z) := \langle z|\,\mathcal{B}\,|z\rangle\,, \tag{1.5}$$

which uniquely determine the operator $\mathcal{B}$.

The trace of the operator $\mathcal{B}$ may be calculated by

$$\begin{aligned}
\mathrm{Tr}(\mathcal{B}) &= \sum_n \langle n|\,\mathcal{B}\,|n\rangle \\
&= \int d^2z \int d^2z' \sum_n \langle n|\,|z\rangle\,\langle z|\,\mathcal{B}\,|z'\rangle\,\langle z'|\,|n\rangle \\
&= \int d^2z \int d^2z' \sum_n e^{-\frac{1}{2}|z|^2-\frac{1}{2}|z'|^2}\frac{1}{n!} z^n z'^n\,\langle z|\,\mathcal{B}\,|z'\rangle \\
&= \int d^2z \int d^2z' e^{-\frac{1}{2}|z|^2-\frac{1}{2}|z'|^2+z\bar{z'}}\,\langle z|\,\mathcal{B}\,|z'\rangle \\
&= \int d^2z \int d^2z'\,\langle z'|\,|z\rangle\,\langle z|\,\mathcal{B}\,|z'\rangle \\
&= \int d^2z\,\langle z|\,\mathcal{B}\,|z\rangle \\
&= \int b(z)d^2z.
\end{aligned}$$

Here in the second line we used the resolution of the identity (1.3) twice and again to get to the last line.

We look at the operators that admit diagonal representation:

$$\mathcal{B} = \pi^{-1}\int B(z)\,|z\rangle\,\langle z|\,d^2z, \tag{1.6}$$

where $B(z)$ is called **the upper symbol** of the operator $\mathcal{B}$.

Example 1.1. *Consider the polynomial in creation and annihilation operators*

$$\mathcal{B} = \sum_{m,n} d_{m,n} a^m a^{*n}.$$

Then

$$\begin{aligned}\mathcal{B} &= \pi^{-1}\int\sum_{m,n} d_{m,n}a^m\,|z\rangle\,\langle z|\,{a^*}^n d^2z\\ &= \pi^{-1}\int[\sum_{m,n} d_{m,n}z^m\bar{z}^n]\,|z\rangle\,\langle z|\,d^2z\\ &= \pi^{-1}\int B(z)\,|z\rangle\,\langle z|\,d^2z,\end{aligned}$$

where we take $B(z)=\sum_{m,n} d_{m,n}z^m\bar{z}^n$.

It follows that

$$\begin{aligned}b(z') &= \langle z'|\,\mathcal{B}\,|z'\rangle\\ &= \pi^{-1}\int B(z)|\,\langle z'|\,|z\rangle\,|^2 d^2z.\end{aligned}$$

From (1.2) we have

$$\begin{aligned}|\,\langle z'|\,|z\rangle\,|^2 &= e^{-|z|^2-|z'|^2+2\,\mathrm{Re}\,\bar{z}'z}\\ &= e^{-|z'-z|^2}.\end{aligned}$$

Therefore

$$b(z)=\pi^{-1}\int B(z)e^{-|z'-z|^2}d^2z$$

The trace of $\mathcal{B}$ may be represented by

$$\begin{aligned}\mathrm{Tr}\mathcal{B} &= \pi^{-1}\int b(z)d^2z\\ &= \pi^{-1}\int d^2z\pi^{-1}\int d^2z'B(z')e^{-|z'-z|^2}\\ &= \pi^{-1}\int d^2z'B(z')\pi^{-1}\int d^2ze^{-|z'-z|^2}\\ &= \pi^{-1}\int B(z)d^2z.\end{aligned}$$

To get to the last line we calculated the integral $\pi^{-1}\int d^2ze^{-|z'-z|^2}=\delta(z-z')$.

And more generally,

$$\begin{aligned}\mathrm{Tr}\mathcal{A}\mathcal{B} &= \pi^{-1}\int\langle z|\,\mathcal{A}\mathcal{B}\,|z\rangle\\ &= \pi^{-1}\int d^2z\pi^{-1}\int d^2z'A(z')\,\langle z|\,|z'\rangle\,\langle z'|\,\mathcal{B}\,|z\rangle\\ &= \pi^{-1}\int A(z)b(z)d^2z\\ \text{and similarly } &= \pi^{-1}\int a(z)B(z)d^2z.\end{aligned}$$

Further reading [1].

2. Bloch Coherent States

The Bloch coherent states (also called Spin coherent states) are similar to the canonical coherent states. Their definition uses the angular momentum operators. We consider a single quantum spin of fixed total angular-momentum J, $J = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and shall denote by $\mathbf{S} = (S_x, S_y, S_z)$ the usual angular momentum operators with commutation relations:

$$[S_x, S_y] = iS_z, \text{ and cyclically.}$$

And

$$\mathbf{S}^2 = S_x^2 + S_y^2 + S_z^2 = J(J+1)I.$$

Define

$$S_\pm = S_x \pm iS_y,$$

then

$$[S_z, S_\pm] = \pm S_\pm, \text{ and } [S_+, S_-] = S_z.$$

The Hilbert space on which these operators act is $\mathbb{C}^{2J+1}$.

We denote by $\mathfrak{S}$ the unit sphere in three dimensions:

$$\mathfrak{S} = \{(x, y, z) | x^2 + y^2 + z^2 = 1\},$$

and by $L^2(\mathfrak{S})$ the space of square integrable function on $\mathfrak{S}$ with the measure

$$\Omega = (\theta, \phi),\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi, \tag{2.1}$$

$$d\Omega = \sin\theta d\theta d\phi, \tag{2.2}$$

$$x = \sin\theta\cos\phi,\ y = \sin\theta\sin\phi,\ z = \cos\theta.$$

The eigenstates of S_z are

$$S_z |s\rangle = s |s\rangle,\ s = -J, \ldots, J.$$

Let us chose $s = J$ then the state $|J\rangle \in \mathbb{C}^{2J+1}$ satisfies

$$S_z |J\rangle = J |J\rangle$$

and

$$S_+ |J\rangle = 0 \text{ and } S_- |J\rangle = |J-1\rangle.$$

Define the Bloch state $|\Omega\rangle \in \mathbb{C}^{2J+1}$ by

$$\begin{aligned} |\Omega\rangle &= e^{\frac{1}{2}\theta e^{i\phi}S_- - \frac{1}{2}\theta e^{-i\phi}S_+} |J\rangle \\ &= e^{zS_-} e^{-\ln(1+|z|^2)S_z} e^{-\bar{z}S_+} |J\rangle, \end{aligned} \tag{2.3}$$

where $z = \tan\frac{\theta}{2}e^{i\phi}$. See Appendix on how to get to the second line. Taking into account that S_+ annihilates $|J\rangle$ we obtain

$$\begin{aligned} |\Omega\rangle &= (1+|z|^2)^{-J} e^{zS_-} |J\rangle \\ &= (1+|z|^2)^{-J} \sum_{n=0}^{2J} \frac{z^n}{n!} |J-n\rangle \\ &= (1+|z|^2)^{-J} \sum_{m=-J}^{J} z^{J-m} \frac{|m\rangle}{(J-m)!} \\ &= (1+|z|^2)^{-J} \sum_{M=-J}^{J} z^{J-M} \binom{2J}{J+M}^{1/2} |M\rangle, \end{aligned} \tag{2.4}$$

where $|M\rangle$ is the normalized state

$$|M\rangle = \binom{2J}{J+M}^{-1/2} \frac{1}{(J-M)!} S_-^{J-M} |J\rangle \tag{2.5}$$

such that

$$S_z |M\rangle = M |M\rangle .$$

Writing z in terms of θ and ϕ we obtain

$$|\Omega\rangle = \sum_{M=-J}^{J} \binom{2J}{J+M}^{1/2} \left(\cos\frac{\theta}{2}\right)^{J+M} \left(\sin\frac{\theta}{2}\right)^{J-M} e^{i(J-M)} |M\rangle .$$

2.1. **Eigenpropoerties of $|\Omega\rangle$.** Note that the operator

$$R_{\theta,\phi} = e^{\frac{1}{2}\theta e^{i\phi} S_- - \frac{1}{2}\theta e^{-i\phi} S_+}$$

is a rotation through an angle θ about an axis $n = (\sin\phi, -\cos\phi, 0)$.

Since $|J\rangle$ is an eigenstate of S_z, $|\Omega\rangle$ is an eigenstate of the rotated S_z

$$(R_{\theta,\phi} S_z R^{-1}_{\theta,\phi}) |\Omega\rangle = J |\Omega\rangle , \tag{2.6}$$

since $|\Omega\rangle = R_{\theta,\phi} |J\rangle$.

The overlap between two Bloch states is given by

$$\begin{aligned} K_J(\Omega', \Omega) &= \langle \Omega' | \, |\Omega\rangle \\ &= (\cos\frac{\theta}{2}\cos\frac{\theta'}{2} + e^{i(\phi-\phi')} \sin\frac{\theta}{2}\sin\frac{\theta'}{2})^{2J}. \end{aligned} \tag{2.7}$$

In particular $|\Omega\rangle$ is normalized since $K_J(\Omega, \Omega) = 1$.

We also have

$$\begin{aligned} &|\cos\frac{\theta}{2}\cos\frac{\theta'}{2} + e^{i(\phi-\phi')} \sin\frac{\theta}{2}\sin\frac{\theta'}{2})|^2 \\ &= (\cos\frac{\theta}{2}\cos\frac{\theta'}{2} + \cos(\phi-\phi')\sin\frac{\theta}{2}\sin\frac{\theta'}{2}))^2 + (\sin(\phi-\phi')\sin\frac{\theta}{2}\sin\frac{\theta'}{2}))^2 \\ &= \frac{1 + \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')}{2} \\ &= \frac{1+\cos\Phi}{2} = \cos^2\frac{1}{2}\Phi, \end{aligned}$$

where $\cos\Phi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$ is the cosine of the angle between Ω and Ω'. Therefore

$$|K_J(\Omega', \Omega)|^2 = \left(\cos\frac{1}{2}\Phi\right)^{4J}.$$

2.2. **Upper and lower symbols.** Let $\mathcal{M}^{2J+1}$ be the set of linear transformation on $\mathbb{C}^{2J+1}$. Define the linear transformation on $\mathbb{C}^{2J+1}$ and, for a given $F \in L^1(\mathfrak{S})$, define $A_F \in \mathcal{M}^{2J+1}$ by

$$A_F = \frac{2J+1}{4\pi} \int d\Omega F(\Omega) |\Omega\rangle \langle\Omega| . \tag{2.8}$$

In fact, every operator in $A \in \mathcal{M}^{2J+1}$ can be written in the form (2.8). In particular,

$$\mathbb{1} = \frac{2J+1}{4\pi} \int d\Omega \, |\Omega\rangle \langle\Omega| . \tag{2.9}$$

The function F corresponding to the operator A is called **the upper symbol**. Thus, to every operator A there correspond two functions: $G_A(\Omega)$ and **the lower symbol**

$$g_A(\Omega) = \langle\Omega|\, A\, |\Omega\rangle \tag{2.10}$$

In the next theorem we show how the find the upper symbol for a particular operator.

Theorem 2.1. *Every operator A in spin space can be expanded in the following manner:*

$$A = \frac{2J+1}{4\pi}\int d\Omega F_A(\Omega)\,|\Omega\rangle\,\langle\Omega|\,, \tag{2.11}$$

where $F_A(\Omega) = Tr[A\Delta_J(\Omega)]$ and for $z = \tan\frac{\theta}{2}e^{i\phi}$

(2.12)

$$\Delta_J(\Omega) = \frac{(-1)^{2J}}{(2J+1)!}\sum_{N,K=-J}^{J}|N\rangle\,\langle K|\sqrt{\frac{(J-N)!(J+N)!}{(J-K)!(J+K)!}}\bar{z}^{N-K}\cdot$$
$$\cdot\sum_{m=0}^{J+N}\sum_{r=0}^{J-N}(-1)^{r+m}\binom{J+K}{J+N-m}\binom{J-K}{J-N-r}\frac{(2J+1+m+r)!}{m!r!}\frac{|z|^{2r}}{(1+|z|^2)^{m+r}}.$$

Proof. Define the integral

$$I_{K_0,N_0} = \frac{2J+1}{4\pi}\int d\Omega\,\langle N_0|\,\Delta_J(\Omega)\,|K_0\rangle\,|\Omega\rangle\,\langle\Omega|\,.$$

Observe that from $d\Omega = \sin\theta d\theta d\phi$ (2.2), we have

$$\begin{aligned}
\int d\Omega\frac{z^j\bar{z}^k}{(1+|z|^2)^m} &= \int\Big(\cos\frac{\theta}{2}\Big)^{2m-j-k}\Big(\sin\frac{\theta}{2}\Big)^{j+k}\sin\theta d\theta\int e^{i\phi(j-k)}d\phi\\
&= 2\pi\delta_{j,k}\int\Big(\cos\frac{\theta}{2}\Big)^{2(m-j)}\Big(\sin\frac{\theta}{2}\Big)^{2j}d(\cos\theta)\\
&= 2\pi\delta_{j,k}\int\Big(\frac{1+\cos\theta}{2}\Big)^{m-j}\Big(\frac{1+\sin\theta}{2}\Big)^{j}d(\cos\theta)\\
&= 4\pi\delta_{j,k}\int_0^1 x^{m-j}(1-x)^j dx\\
&= 4\pi\delta_{j,k}\frac{j!(m-j)!}{(m+1)!}. \qquad (2.13)
\end{aligned}$$

From the definition of the Bloch coherent state the projection can be written (2.14)

$$|\Omega\rangle\,\langle\Omega| = \frac{1}{(1+|z|^2)^{2J}}\sum_{M,L=-J}^{J}\binom{2J}{J+M}^{1/2}\binom{2J}{J+L}^{1/2}z^{J-M}\bar{z}^{J-L}\,|M\rangle\,\langle L|\,.$$

Therefore from (2.13) and (2.12) we get

$$
\begin{aligned}
I_{K_0,N_0} &= \frac{(-1)^{2J}}{4\pi(2J)!}\sqrt{\frac{(J-N_0)!(J+N_0)!}{(J-K_0)!(J+K_0)!}}\sum_{M,L=-J}^{J}|M\rangle\langle L|\cdot \\
&\cdot \sum_{m=0}^{J+N_0}\sum_{r=0}^{J-N_0}(-1)^{r+m}\binom{J+K_0}{J+N_0-m}\binom{J-K_0}{J-N_0-r}\frac{(2J+1+m+r)!}{m!r!}\binom{2J}{J+M}^{1/2}\cdot \\
&\cdot \binom{2J}{J+L}^{1/2}\int d\Omega\frac{z^{J-M+r}\bar{z}^{J-L+N-K+r}}{(1+|z|^2)^{m+r+2J}} \\
&= \frac{(-1)^{2J}}{4\pi(2J)!}\sqrt{\frac{(J-N_0)!(J+N_0)!}{(J-K_0)!(J+K_0)!}}\sum_{M,L=-J}^{J}|M\rangle\langle L|\cdot \\
&\cdot \sum_{m=0}^{J+N_0}\sum_{r=0}^{J-N_0}(-1)^{r+m}\binom{J+K_0}{J+N_0-m}\binom{J-K_0}{J-N_0-r}\frac{(2J+1+m+r)!}{m!r!}\binom{2J}{J+M}^{1/2}\cdot \\
&\cdot \binom{2J}{J+L}^{1/2}4\pi\delta_{M,K_0-N_0+L}\frac{(J+M+m)!(J-M+r)!}{(2J+1+M+r)!} \\
&= (-1)^{2J}\sqrt{\frac{(J-N_0)!(J+N_0)!}{(J-K_0)!(J+K_0)!}}\sum_{M,L=-J}^{J}\frac{\delta_{M,K_0-N_0+L}|M\rangle\langle L|}{\sqrt{(J+M)!(J-M)!(J+L)!(J-L)!}}\cdot \\
&\cdot \sum_{m=0}^{J+N_0}\sum_{r=0}^{J-N_0}(-1)^{r+m}\binom{J+K_0}{J+N_0-m}\binom{J-K_0}{J-N_0-r}\frac{(J+M+m)!(J-M+r)!}{m!r!}.
\end{aligned}
$$

Note that there is a relation between binomial coefficients

$$
\sum_{r=0}^{N}(-1)^r\binom{k}{N-r}\frac{(n+r)!}{r!}=n!\binom{k-n-1}{N}.
$$

Therefore using this relation for r and m we obtain

$$
\begin{aligned}
I_{K_0,N_0} &= (-1)^{2J}\sqrt{\frac{(J-N_0)!(J+N_0)!}{(J-K_0)!(J+K_0)!}}\sum_{M,L=-J}^{J}\delta_{M,K_0-N_0+L}|M\rangle\langle L| \\
&\cdot\sqrt{\frac{(J-M)!(J+M)!}{(J-L)!(J+L)!}}\binom{N_0-L-1}{J+N_0}\binom{L-N_0-1}{J-N_0},
\end{aligned}
$$

where we used the fact that $M-K_0=L-N_0$ from the Kronecker symbol.

Since

$$
\binom{N_0-L-1}{J+N_0}\binom{L-N_0-1}{J-N_0}=(-1)^{2J}\delta_{N_0,L},
$$

we have

$$
\frac{2J+1}{4\pi}\int d\Omega\,\langle N_0|\,\Delta_J(\Omega)\,|K_0\rangle\,|\Omega\rangle\,\langle\Omega| = |N_0\rangle\,\langle K_0|\,.
$$

And from here

$$
\delta_{K,K_0}\delta N,N_0=\frac{2J+1}{4\pi}\int d\Omega\,\langle N_0|\,\Delta_J(\Omega)\,|K_0\rangle\,\langle K|\,|\Omega\rangle\,\langle\Omega|\,|N\rangle\,.
$$

Multiplying both sides by $|N_0\rangle\langle K_0|$ and summing it over N_0, K_0

$$
\begin{aligned}
|N\rangle\langle K| &= \sum_{N_0,K_0} \delta_{K,K_0}\delta N, N_0 |N_0\rangle\langle K_0| \\
&= \sum_{N_0,K_0} \frac{2J+1}{4\pi}\int d\Omega \langle N_0|\Delta_J(\Omega)|K_0\rangle\langle K||\Omega\rangle\langle\Omega||N\rangle|N_0\rangle\langle K_0| \\
&= \frac{2J+1}{4\pi}\int d\Omega \langle K||\Omega\rangle\langle\Omega||N\rangle\Delta_J(\Omega),
\end{aligned}
$$

since every operator can be written as a sum of its matrix elements

$$A = \sum_{N,K=-J}^{J} \langle N|A|K\rangle|N\rangle\langle K|.$$

Therefore

$$|N\rangle\langle K| = \frac{2J+1}{4\pi}\int d\Omega \mathrm{Tr}[|N\rangle\langle K|\Delta_J(\Omega)]|\Omega\rangle\langle\Omega|$$

and so

$$A = \frac{2J+1}{4\pi}\int d\Omega \mathrm{Tr}[A\Delta_J(\Omega)]|\Omega\rangle\langle\Omega|.$$

The proof is based on [4]. □

2.3. **Table.** In the following table we list some function and their upper and lower symbols

Operator	$g(\Omega)$	$G(\Omega)$
S_z	$J\cos\theta$	$(J+1)\cos\theta$
S_x	$J\sin\theta\cos\phi$	$(J+1)\sin\theta\cos\phi$
S_y	$J\sin\theta\sin\phi$	$(J+1)\sin\theta\cos\phi$
S_z^2	$J(J-\frac{1}{2})(\cos\theta)^2+\frac{J}{2}$	$(J+1)(J+\frac{3}{2})(\cos\theta)^2-\frac{1}{2}(J+1)$

2.4. **Remarks.** Some final remarks. First, if we consider $|\Omega'\rangle\langle\Omega| \in \mathcal{M}^{2J+1}$ then as may be seen from (2.4)

$$
\begin{aligned}
\mathrm{Tr}|\Omega\rangle\langle\Omega'| &= \sum_{M=-J}^{J}\langle M||\Omega\rangle\langle\Omega'||M\rangle \\
&= \langle\Omega'||\Omega\rangle = K_J(\Omega',\Omega). \quad (2.15)
\end{aligned}
$$

Hence from (2.8)

$$\mathrm{Tr}A_G = \frac{2J+1}{4\pi}\int d\Omega G(\Omega). \quad (2.16)$$

The second remark is that from the resolution of the identity (2.9) and the definition of K_J (2.7)

$$\frac{2J+1}{4\pi}\int d\Omega K_J(\Omega',\Omega)K_J(\Omega,\Omega'') = K_J(\Omega',\Omega''). \quad (2.17)$$

Thus, K_J reproduces itself under convolution.

The third remark is that for any $A \in \mathcal{M}^{2J+1}$ we can use the resolution of identity (2.9) to obtain

$$
\begin{aligned}
\mathrm{Tr}A &= \frac{2J+1}{4\pi}\int d\Omega \mathrm{Tr}\,|\Omega\rangle\langle\Omega|\,A \\
&= \frac{2J+1}{4\pi}\int d\Omega \sum_{M=-J}^{J}\langle M|\,|\Omega\rangle\langle\Omega|\,A\,|M\rangle \\
&= \frac{2J+1}{4\pi}\int d\Omega\,\langle\Omega|\,A\,|\Omega\rangle \\
&= \frac{2J+1}{4\pi}\int g_A(\Omega)d\Omega.
\end{aligned} \tag{2.18}
$$

3. Lower bound to the quantum partition function

We consider a system of N quantum spins. The Hilbert space is

$$\mathcal{H}_N = \bigotimes_{i=1}^{N}\mathcal{H}^i = \bigotimes_{i=1}^{N}\mathbb{C}^{2J^i+1}.$$

The Hamiltonian, H, can be general, but can always be written as a polynomial in the $3N$ spin operators.

The partition function is

$$Z^Q = \alpha_N \mathrm{Tr}e^{-\beta H}, \tag{3.1}$$

where $\alpha_N = \prod_{i=1}^{N}(2J^i+1)^{-1}$ is a normalization factor, which is not essential here.

We denote by

$$|\Omega_N\rangle = \bigotimes_{i=1}^{N}\left|\Omega^i\right\rangle \tag{3.2}$$

the complete, normalized set of states on $\mathcal{H}_N$.

Using (2.18),

$$Z^Q = (4\pi)^{-N}\int d\Omega_N\,\langle\Omega_N|\,e^{-\beta H}\left|\Omega^N\right\rangle.$$

By the Peierls-Bogoliubov inequality

$$\langle\phi|\,e^X\,|\phi\rangle \geq exp\,\langle\phi|\,X\,|\phi\rangle$$

for any normalized $\phi \in \mathcal{H}_N$ and X selfadjoint. Thus,

$$Z^Q \geq (4\pi)^{-N}\int d\Omega_N exp\{-\beta\,\langle\Omega_N|\,H\,|\Omega_N\rangle\}. \tag{3.3}$$

We suppose that the Hamiltonian is H is linear in the operators $\mathbf{S}^i$ of each spin. That is we allow multiple site interactions of arbitrary complexity such as $S_x^1 S_y^2 s_y^3 S_z^4$, but do not allow monomials such as $(S_x^1)^2$ or $S_x^1 S_y^1$. We shall refer to this case as **the normal case**.

From the definition of the lower symbol (2.10) and the table of the lower symbols the right hand side of (3.3) is precisely the classical partition function in which each $\mathbf{S}^i$ is replaces by J^i times a unit vector in $\mathfrak{S}$. I.e.

$$\mathbf{S}^i \to J^i(\sin\theta^i\cos\phi^i, \sin\theta^i\sin\phi^i, \cos\theta^i). \tag{3.4}$$

Thus in the normal case,

$$Z^Q \geq Z^C(J^1, ...J^N), \tag{3.5}$$

where Z^C means the classical partition function for the classical Hamiltonian obtained from the quantum one by the replacing each angular momentum operator $\mathbf{S}^i$ by J^i times the unit vector (3.4).

4. Upper bound to the quantum partition function

From the definition of the partition function (3.1)

$$Z^Q = \lim_{n\to\infty} Z(n), \tag{4.1}$$

where

$$Z(n) = \alpha_N \mathrm{Tr}(\mathbb{1} - \beta n^{-1} H)^n. \tag{4.2}$$

Now denote the upper symbol of the Hamiltonian H by $G(\Omega_N)$, so $(1 - \frac{1}{n}\beta H)$ is represented by

$$F_n(\Omega_N) = 1 - \beta n^{-1} G(\Omega_N). \tag{4.3}$$

Therefore by (2.8)

$$(1-\beta n^{-1}H)^n = C\int d\Omega_{N^1}...\int d\Omega_{N^n}\prod_{j=1}^{n} F_n(\Omega_{N^j})\left|\Omega_{N^1}\right\rangle\left\langle\Omega_{N^1}\right|...\left|\Omega_{N^n}\right\rangle\left\langle\Omega_{N^n}\right|,$$

where C is a normalization constant. So taking the trace

$$\mathrm{Tr}(1-\beta n^{-1}H)^n = C\int d\Omega_{N^1}...\int d\Omega_{N^n}\prod_{j=1}^{n} F_n(\Omega_{N^j})\mathrm{Tr}\Big(\left|\Omega_{N^1}\right\rangle\left\langle\Omega_{N^1}\right|...\left|\Omega_{N^n}\right\rangle\left\langle\Omega_{N^n}\right|\Big).$$

Calculating the trace

$$\begin{aligned}
&\mathrm{Tr}\left|\Omega_{N^1}\right\rangle\left\langle\Omega_{N^1}\right|...\left|\Omega_{N^n}\right\rangle\left\langle\Omega_{N^n}\right| = \mathrm{Tr}\Big(\bigotimes_{i=1}^{N}\left|\Omega^i_{N^1}\right\rangle\left\langle\Omega^i_{N^1}\right|...\left|\Omega^i_{N^n}\right\rangle\left\langle\Omega^i_{N^n}\right|\Big)\\
&= \prod_{i=1}^{N}\mathrm{Tr}\left|\Omega^i_{N^1}\right\rangle\left\langle\Omega^i_{N^1}\right|...\left|\Omega^i_{N^n}\right\rangle\left\langle\Omega^i_{N^n}\right| = \prod_{i=1}^{N}\prod_{j=1}^{n} K_{J^i}(\Omega^i_j, \Omega^i_{j+1})\\
&= \prod_{j=1}^{n} L_J(\Omega^j_N, \Omega^{j+1}_N),
\end{aligned}$$

where we defined

$$L_J(\Omega^j_N, \Omega^{j+1}_N) = \prod_{i=1}^{N} K_{J^i}(\Omega^i_j, \Omega^i_{j+1}). \tag{4.4}$$

To get the first equality we used the definition of $|\Omega_N\rangle$ (3.2) and to get to the third line we used the definition of K_J (2.7) and (2.15).

So $Z(n)$ can be represented as an nN-fold integral

$$Z(n) = \alpha_N\int d\Omega_{N^1}...\int d\Omega_{N^n}\prod_{j=1}^{n} F_n(\Omega_{N^j})L_J(\Omega_{N^j}, \Omega_{N^{j+1}}), \tag{4.5}$$

with $n+1 = 1$ in the last factor, and where $L_J(\Omega_{N'}, \Omega_N)$ is defined in (4.4).

From the definition of L_J (4.4)

$$L_J(\Omega_{N'}, \Omega_N) = (4\pi)^{-N} \alpha_N^{-1} \tag{4.6}$$

and from the convolution property for K_J (2.17)

$$\int d\Omega_N L_J(\Omega_{N'}, \Omega_N) L_J(\Omega_N, \Omega_{N''}) = L_J(\Omega_{N'}, \Omega_{N''}). \tag{4.7}$$

Now we think of F_n as a multiplication operator and of L_J as an integral kernel of a compact self-adjoint operator on $L^2(\mathfrak{S})$ in (4.5).

Let B be the operator with an integral kernel $B(\Omega_{N^1}, \Omega_{N^2})$, then

$$B(f)(\Omega_{N_1}) = \int B(\Omega_{N_1}, \Omega_{N_2}) f(\Omega_{N_2}) d\Omega_{N_2},$$

for any $f \in L^1(\mathfrak{S})$. And

$$\mathrm{Tr} B = \int d\Omega_N B(\Omega_N, \Omega_N). \tag{4.8}$$

The operator B^2 can be calculated

$$\begin{aligned} B^2(f)(\Omega_{N_1}) &= B(B(f))(\Omega_{N_1}) = \int B(\Omega_{N_1}, \Omega_{N_2}) B(f)(\Omega_{N_2}) d\Omega_{N_2} \\ &= \int B(\Omega_{N_1}, \Omega_{N_2}) \int B(\Omega_{N_2}, \Omega_{N_3}) f(\Omega_{N_3}) d\Omega_{N_3} d\Omega_{N_2} \\ &= \int \Big[\int B(\Omega_{N_1}, \Omega_{N_2}) B(\Omega_{N_2}, \Omega_{N_3}) d\Omega_{N_2} \Big] f(\Omega_{N_3}) d\Omega_{N_3}, \end{aligned}$$

so operator B^2 has kernel

$$B^2(\Omega_{N^1}, \Omega_{N^3}) = \int d\Omega_{N^2} B(\Omega_{N^1}, \Omega_{N^2}) B(\Omega_{N^2}, \Omega_{N^3}).$$

Similarly operator B^n has kernel

$$B^n(\Omega_{N^1}, \Omega_{N^{n+1}}) = \int d\Omega_{N^2} ... d\Omega_{N^n} B(\Omega_{N^1}, \Omega_{N^2}) ... B(\Omega_{N^n}, \Omega_{N^{n+1}}). \tag{4.9}$$

Then from (4.5) and (4.8) Z_n can be written

$$Z(n) = \alpha_N \mathrm{Tr}(F_n L_J)^n. \tag{4.10}$$

In general if $m = 2^j$, $j = 0, 1, 2, ...$ the following inequality can be proven by induction (see [5])

$$|\mathrm{Tr}(AB)^{2m}| \leq \mathrm{Tr} A^{2m} B^{2m} \tag{4.11}$$

whenever A and B are self-adjoint operators.

Hence if we take $n = 2^j$ in the the definition of $Z(n)$ (4.2) and use (4.11), we obtain

$$\begin{aligned}
Z(n) &\leq \alpha_N \mathrm{Tr}(F_n^n L_J^n) \\
&= \alpha \int d\Omega_N F_n^n(\Omega_N) L_J^n(\Omega_N, \Omega_N) \\
&= \alpha \int d\Omega_N F_n^n(\Omega_N) \int d\Omega_{N^2} ... d\Omega_{N^n} L_J(\Omega_N, \Omega_{N^2}) ... L_J(\Omega_{N^n}, \Omega_N) \\
&= \alpha \int d\Omega_N F_n^n(\Omega_N) L_J(\Omega_N, \Omega_N) \\
&= \alpha \int d\Omega_N F_n^n(\Omega_N) (4\pi)^{-N} \alpha_N^{-1} \\
&= (4\pi)^{-N} \int d\Omega_N F_n^n(\Omega_N) \\
&= (4\pi)^{-N} \int d\Omega_N (1 - \beta n^{-1} G(\Omega_N))^n.
\end{aligned}$$

where in the second equality we used (4.9), in the third equality we used (4.7) n times and in the fourth equality we used (4.6) and we used the definition of F_n (4.3) in the last equality.

Therefore in the limit $n \to \infty$ we get

$$Z^Q \leq (4\pi)^{-N} \int d\Omega_N \exp[-\beta G(\Omega_N)]. \tag{4.12}$$

In the normal case, when the Hamiltonian H is linear in each $\mathbf{S}^i$, the upper symbol $G(\Omega_N)$ replaces each $\mathbf{S}^i$ by $(J^i + 1)$ times a unit vector in $\mathfrak{S}$. Thus

$$Z^Q \leq Z^C(J^1 + 1, ... J^N + 1). \tag{4.13}$$

And putting the upper (4.13) and lower (3.5) bounds together we get the relationship between the quantum and classical partition functions for the normal case Hamiltonian

$$Z^C(J^1, ... J^N) \leq Z^Q \leq Z^C(J^1 + 1, ... J^N + 1). \tag{4.14}$$

5. Thermodynamics limit

We consider here only the normal case. Let H_N be a Hamiltonian of N spins in which each spin has angular momentum one. Replace each spin operator $\mathbf{S}^i$ by $J^{-1}\mathbf{S}^i$ and let $\mathbf{S}^i$ now have angular momentum J. We shall denote the resulting Hamiltonian by $H_N^Q(J)$ and the corresponding quantum partition function by $Z_N^Q(J)$.

For both quantum and classical partition function $Z_N(J)$ we denote the free energy per spin by

$$f_N(J) = -(N\beta)^{-1} \ln Z_N(J).$$

Therefore we have the original Hamiltonian H_N^Q which is a polynomial in $\mathbf{S}^i$ each of which has angular momentum 1. Replacing each $\mathbf{S}^i$ by a unit vector u^i in $\mathfrak{S}$ we obtain the classical Hamiltonian H_N^C and corresponding partition function Z_N^C and the free energy per spin f_N^C.

From the quantum Hamiltonian $H_N^Q(J)$, that we constructed above as a polynomial in $J^{-1}\mathbf{S}^i$, we get the classical Hamiltonian by replacing each spin operator $\mathbf{S}^i$ by $\tilde{J}u^i$, for $u^i \in \mathfrak{S}$ and any $\tilde{J}$. So the classical Hamiltonian becomes a polynomial in $\frac{J+1}{J}u^i$, denote $\delta_J = \frac{J+1}{J}$. Then the classical Hamiltonian is denoted by $H_N^C(\delta) = Polynom(\delta u^i)$. The corresponding partition function is denoted by $Z_N^C(J+1)$ and the free energy function is denoted by $f_N^C(\delta_J)$. Note that $H_N^C(1) = H_N^C$ and $f_N^C(1) = f_N^C$.

Theorem 5.1. *For a particular class of Hamiltonians (see Assumption 1 below) and with the above construction of the classical free energy function in mind, the following inequality holds*

$$\lim_{J\to\infty}\lim_{N\to\infty} f_N^Q(J) = f^C = \lim_{N\to\infty} f_N^C. \tag{5.1}$$

Proof. From the previous sections (4.14) we know that

$$f_N^C \geq f_N^Q(J) \geq f_N^C(\delta_J). \tag{5.2}$$

Now we think of δ_J as a variable δ. Then $\lim_{J\to\infty}$ is the same as $\lim_{\delta\to 1}$. The classical Hamiltonian $H_N^C(\delta)$ is continuous in δ since it is a polynomial in δ.

We consider a class of Hamiltonians such that $N^{-1}H_N^C(\delta)$ is equicontinuous in N, i.e.

$$\forall\epsilon > 0\ \exists\gamma > 0 \text{ such that } \forall N:\ \|H_N^C(\delta + x) - H_N^C(\delta)\| \leq N\epsilon,$$

whenever $|x| < \gamma$ (γ is independent of N). The norm is taken the uniform norm on $\mathfrak{S}_N$.

For example, it is enough to assume that:

Assumption 1. *For every N we assume that the normal case Hamiltonian H_N satisfies the following two conditions*

- *the degree of the Hamiltonian (which is a polynomial in a normal case) do not exceed some fixed number d and*
- *the sum of coefficients in H_N is no greater than N (possibly, times some fixed constant).*

For example, one cane take the Heisenberg Hamiltonian $H_N^Q = \sum_{i=1}^N \mathbf{S}^i\mathbf{S}^{i+1}$.

The uniform norm looks like $\|H_N^C(\delta)\| = \sup_{\Omega^i}|Polynom(\delta\Omega^i)|$. Since

$$|Polynom((\delta + x)\Omega^i) - Polynom(\delta\Omega^i)| \leq |x|d(\delta+1)^{d-1}N, \tag{5.3}$$

the norm can be bounded

$$\|H_N^C(\delta + x) - H_N^C(\delta)\| \leq N\epsilon$$

whenever $|x| < \gamma < \frac{\epsilon}{d(\delta+1)^{d-1}}$. Therefore $N^{-1}H_N^C$ is equicontinuous in N.

Note that for $|x| < \gamma$, any Ω_N and any value of any Ω^i

$$|Polynom(\delta\Omega^i)| \leq \delta^d N,$$

so

$$e^{-\beta\delta^d N} \leq e^{-\beta Polynom(\delta\Omega^i)} \leq e^{\beta\delta^d N}.$$

Then

$$|\ln Z_N^C(\delta)| = |\ln(4\pi)^{-N}\int d\Omega_N e^{-\beta Polynom(\delta\Omega^i)}| \leq N\beta\delta^d \tag{5.4}$$

and therefore there is a constant K such that for all N the uniform norm of the free energy function is bounded above

$$|f_N^C(\delta)| \leq K.$$

So f_N^C is uniformly bounded.

To show the equicontinuity of the free energy we use the continuity of the logarithm function. From continuity of logarithm, for the inequality to hold

$$|\ln Z_N^C(\delta + x) - \ln Z_N^C(\delta)| \leq N\beta\epsilon,$$

we need the following inequality to be true

$$|Z_N^C(\delta + x) - Z_N^C(\delta)| \leq |Z_N^C(\delta)| \min\{e^{N\beta\epsilon} - 1, 1 - e^{-N\beta\epsilon}\}. \tag{5.5}$$

The left hand side in (5.5) is bounded above similarly to (5.3) by

$$|Z_N^C(\delta + x) - Z_N^C(\delta)| \leq |x| N\beta d(\delta + 1)^d e^{-N\beta(\delta+1)^d}.$$

The right hand side of (5.5) is bounded below using (5.4) by

$$e^{-N\beta\delta^d} \min\{e^{\beta\epsilon} - 1, 1 - e^{-\beta\epsilon}\}.$$

Therefore from (5.5) we have

$$|x| \leq \frac{1}{\beta d(\delta + 1)^d} \frac{e^{N\beta((\delta+1)^d - \delta^d)}}{N} \min\{e^{\beta\epsilon} - 1, 1 - e^{-\beta\epsilon}\}.$$

Since $\frac{e^{N\beta((\delta+1)^d - \delta^d)}}{N} \to \infty$ as $N \to \infty$, for any $\epsilon > 0$ there exists $\gamma > 0$ independent of N such that for any N

$$|f_N^C(\delta + x) - f_N^C(\delta)| < \epsilon$$

whenever $|x| < \gamma$.

So $f_N^C(\delta)$ is equicontinuous in N.

Hence, by Arzela-Ascoli theorem, the limit function

$$f^C(\delta) = \lim_{N\to\infty} f_N^C(\delta) \tag{5.6}$$

exists and is continuous in δ. Therefore

$$\begin{aligned} &\lim_{J\to\infty} \lim_{N\to\infty} f_N^C(\delta_J) = \lim_{\delta\to 1} \lim_{N\to\infty} f_N^C(\delta) \\ &= \lim_{\delta\to 1} f^C(\delta) =: f^C. \end{aligned} \tag{5.7}$$

Taking $\delta = 1$ in (5.6)

$$\lim_{J\to\infty} \lim_{N\to\infty} f_N^C = \lim_{N\to\infty} f_N^C = f^C, \tag{5.8}$$

then by (5.2), (5.7) and (5.8) we have

$$\lim_{J\to\infty} \lim_{N\to\infty} f_N^C(J) = f^C = \lim_{N\to\infty} f_N^C.$$

□

Further reading [3].

6. Appendix

Consider 2×2 matrix representation:

$$S_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ S_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ S_z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

We would like to find the relationship between ω's and x's in the formula

$$e^{\omega_+ S_+ + \omega_- S_- + \omega_z S_z} = e^{x_- S_-} e^{\ln x_z S_z} e^{x_+ S_+}.$$

Writing the exponent as a series

$$e^{\omega_+ S_+ + \omega_- S_- + \omega_z S_z} = \sum_n \frac{1}{n!}(\omega_+ S_+ + \omega_- S_- + \omega_z S_z),$$

define $A := \omega_+ S_+ + \omega_- S_- + \omega_z S_z$. To find the n-th power of A we first diagonalize the operator

$$A = \begin{pmatrix} \frac{1}{2}\omega_z & \omega_+ \\ \omega_- & \frac{1}{2}\omega_z \end{pmatrix}.$$

The eigenvalues of A are $\lambda = \pm K = \pm(\omega_+\omega_- + \frac{1}{4}\omega_z^2)^{1/2}$ and eigenvectors are $v_{1,2} = \begin{pmatrix} \frac{1}{2}\omega_z \pm K \\ \omega_- \end{pmatrix}$.

Then $A = VDV^{-1}$, where

$$V = \begin{pmatrix} \frac{1}{2}\omega_z + K & \frac{1}{2}\omega_z - K \\ \omega_- & \omega_- \end{pmatrix} \text{ and } D = \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix}.$$

Then $A^n = VD^nV^{-1}$. Multiplying three matrices

$$A^n = \begin{pmatrix} \frac{K^n+(-K)^n}{2} + \frac{1}{2}\omega_z \frac{K^n-(-K)^n}{2K} & \omega_+ \frac{K^n-(-K)^n}{2K} \\ \omega_- \frac{K^n-(-K)^n}{2K} & \frac{K^n+(-K)^n}{2} - \frac{1}{2}\omega_z \frac{K^n-(-K)^n}{2K} \end{pmatrix}.$$

Therefore

$$\begin{aligned} e^{\omega_+ S_+ + \omega_- S_- + \omega_z S_z} &= \sum_n \frac{1}{n!} A^n \\ &= \begin{pmatrix} \cosh K + \frac{1}{2}\omega_z \frac{\sinh K}{K} & \omega_+ \frac{\sinh K}{K} \\ \omega_- \frac{\sinh K}{2K} & \cosh K - \frac{1}{2}\omega_z \frac{\sinh K}{K} \end{pmatrix}. \end{aligned}$$

Similarly calculating the matrices we find that

$$e^{x_- S_-} e^{\ln x_z S_z} e^{x_+ S_+} = \begin{pmatrix} x_z^{1/2} & x_+ x_z^{1/2} \\ x_- x_z^{1/2} & x_z^{-1/2} + x_+ x_- x_z^{1/2} \end{pmatrix}.$$

Therefore we get the relationship between ω's and x's. In the case of Bloch coherent states (2.3) we get

$$\omega_+ = -\frac{1}{2}\theta e^{-i\phi},\ \omega_- = \frac{1}{2}\theta e^{i\phi},\ \omega_z = 0$$

and

$$x_+ = -\bar{z} = \tan\frac{\theta}{2} e^{-i\phi},\ x_- = z = \tan\frac{\theta}{2} e^{i\phi},\ x_z = (1 + |z|^2)^{-1} = \cos^2\frac{\theta}{2}.$$

The proof is based on [6].

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